October 23, 2024

Technical Appendix B to Accompany
"Increased (Platform) Competition Reduces (Seller) Competition"
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This Technical Appendix expands the analysis from the paper to include multi-homing of sellers, seller entry and competition, and endogenous platform strength. I begin by revisiting the key equations, Lemmas, and Propositions detailed in the paper.

Equations, Lemmas, and Propositions from the Text

$$q_{kj}^P = \theta_j \, \alpha_j - \beta_j^P \, p_{kj}^P + \eta_j \, p_{kj}^S. \tag{1}$$

$$q_{kj}^S = \alpha_j - \beta_j^S \, p_{kj}^S + \eta_j \, p_{kj}^P. \tag{2}$$

$$q_{kj}^S = A_j - b_j^S \, p_{kj}^S. \tag{3}$$

$$\phi_{kj} \equiv \left[\left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,(2+\Omega_j)} \right) \frac{\widetilde{\Delta}_{ij}}{\Delta_{kj}} \right]^2. \tag{4}$$

$$f(r,n) \equiv 2\,[1+\gamma\,(n-2)]\,[2+\gamma\,(n-2)] - \gamma^2\,[n-1]\,. \tag{5}$$

$$g(r,n) \equiv \gamma^2\,[n-1] + 4\,[1+\gamma\,(n-2)]^2\,[2+\gamma\,(n-2)]\,. \tag{6}$$

$$\begin{aligned} H_{kj} \equiv \frac{1}{2} & \left\{ \frac{\overline{\Delta}_{kj}}{2} + \frac{\Delta_{kj}\gamma\,[n-1]\left\{2\,[1+\gamma\,(n-2)]^2 + \gamma^2\,[n-1] + 2\,[1+\gamma\,(n-2)]\,f(r,n)\right\}}{2g(r,n)} \right\} \\ & \cdot \left\{ \frac{\Delta_{kj}\gamma\,[n-1]\left\{2\,[1+\gamma\,(n-2)]^2 + \gamma^2\,[n-1]\right\}}{g(r,n)} + \overline{\Delta}_{kj} \right\} \\ & + \left\{ \Delta_{kj}\frac{\gamma^4\,[n-1]^2 + 4\,[1+\gamma\,(n-2)]^3\,[2+\gamma\,(n-2)]}{2g(r,n)} + \frac{\gamma\overline{\Delta}_{kj}}{2} \right\} \\ & \cdot [\,n-1\,]\,\frac{\Delta_{kj}\,[1+\gamma\,(n-2)]\left\{2\,[1+\gamma\,(n-2)]^2 + \gamma^2\,[n-1]\right\}}{g(r,n)}. \end{aligned} \tag{7}$$

$$\begin{aligned} \varsigma_{kj} \equiv \frac{1}{\beta_j^S\,\beta_j^P\,[1-\Omega_j]} & \left\{ \frac{\eta_j\,[2+\Omega_j]\,\Delta_{kj}}{8+\Omega_j} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j\,(2+\Omega_j)\,\Delta_{kj}}{2\beta_j^S\,(8+\Omega_j)} \right] \right. \\ & \left. + \frac{\beta_j^S}{2}\left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j\,(2+\Omega_j)\,\Delta_{kj}}{2\beta_j^S\,(8+\Omega_j)} \right]^2 + \frac{\beta_j^P}{2}\left[\frac{(2+\Omega_j)\,\Delta_{kj}}{8+\Omega_j} \right]^2 \right\}. \end{aligned} \tag{8}$$

$$\widetilde{\Delta}_{kj} \;=\; \Delta_{kj} + \frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} \;=\; \alpha_j - \beta_j^S\, c_j^S + \eta_j\, c_{kj}^P + \frac{\eta_j}{\beta_j^P}\left[\,\theta_j\,\alpha_j - \beta_j^P\, c_{kj}^P + \eta_j\, c_j^S\,\right]. \tag{9}$$

$$\frac{\partial \widetilde{\Delta}_{kj}}{\partial c_{kj}^P} \;=\; \eta_j - \frac{\eta_j}{\beta_j^P}\,\beta_j^P \;=\; 0. \tag{10}$$

$$b_j^S \;=\; \beta_j^S\left[1 - \frac{\left(\eta_j\right)^2}{\beta_j^S\,\beta_j^P}\right] \;=\; \beta_j^S\,[\,1-\Omega_j\,]. \tag{11}$$

$$\widetilde{\Delta}_{kj} \;=\; \Delta_{kj} + \frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}. \tag{12}$$

(10) and (12) imply that:

$$\widetilde{\Delta}_{1j} \;=\; \widetilde{\Delta}_{2j} \;>\; \Delta_{2j} \text{ and } \widetilde{\Delta}_{2j} \;=\; \widetilde{\Delta}_{1j} \;>\; \Delta_{1j}. \tag{13}$$

Lemma 1. *Suppose Sj faces no competition from Pk when Sj sells on Pk* ($j,k \in \{1,2\}$)*. Given* w_{kj}*, Sj's equilibrium output (i.e., sales)* (Q_{kj}^S) *is* $\frac{\Theta_k\left[\widetilde{\Delta}_{kj} - b_j^S\, w_{kj}\right]}{2}$*, and Sj's total profit is* $\frac{\Theta_k}{b_j^S}\left[\,q_{kj}^S\,\right]^2$ *where* $q_{kj}^S = \frac{Q_{kj}^S}{\Theta_k}$.

Lemma 2. *Suppose Sj faces no competition from Pk when Sj sells on Pk* ($j,k \in \{1,2\}$)*. Then Pk's profit-maximizing commission for Sj is* $w_{kj} = \frac{\widetilde{\Delta}_{kj}}{2\,b_j^S}$.

Lemma 3. *Suppose Sj faces no competition from Pk when Sj sells on Pk* ($j,k \in \{1,2\}$)*. Then Sj's equilibrium output* (Q_{kj}^S) *is* $\frac{\Theta_k\,\widetilde{\Delta}_{kj}}{4}$*, Sj's profit is* $\frac{\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{16\,b_j^S}$*, and Pk's profit from the commission it collects from Sj is* $\frac{\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{8\,b_j^S}$.

Lemma 4. *Suppose Sj competes against Pk when Sj sells on Pk* ($j,k \in \{1,2\}$)*. Given* w_{kj}*, Sj's equilibrium output* (Q_{kj}^S) *is* $\frac{\Theta_k\left[\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj} - 2\,\beta_j^S(\,1-\Omega_j\,)\,w_{kj}\right]}{4-\Omega_j}$*, Pk's equilibrium output* (Q_{kj}^P) *is* $\frac{\Theta_k\left[2\,\overline{\Delta}_{kj} + \frac{\eta_j}{\beta_j^S}\,\Delta_{kj} - \eta_j(\,1-\Omega_j\,)\,w_{kj}\right]}{4-\Omega_j}$*, and Sj's total profit is* $\frac{\Theta_k}{\beta_j^S}\left[q_{kj}^S\right]^2$ *where* $q_{kj}^S = \frac{Q_{kj}^S}{\Theta_k}$.

Lemma 5. *Suppose Sj competes against Pk when Sj sells on Pk* ($j,k \in \{1,2\}$)*. Then Pk's profit-maximizing commission for Sj is*

$$\frac{1}{2[\,1-\Omega_j\,]}\left[\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} + \frac{\left[8+(\Omega_j)^2\right]\Delta_{kj}}{\beta_j^S(\,8+\Omega_j\,)}\right].$$

Lemma 6. *Suppose Sj competes against Pk when Sj sells on Pk* ($j,k \in \{1,2\}$)*. Then Sj's equilibrium output is* $\frac{\Theta_k[\,2+\Omega_j\,]\,\Delta_{kj}}{8+\Omega_j}$*, Sj's profit is* $\frac{\Theta_k}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\,\Delta_{kj}}{8+\Omega_j}\right]^2$*, Pk's equilibrium output is* $\frac{\Theta_k\,\overline{\Delta}_{kj}}{2} + \frac{\eta_j\Theta_k[\,2+\Omega_j\,]\,\Delta_{kj}}{2\beta_j^S[\,8+\Omega_j\,]}$*, and Pk's profit from the commission it collects from Sj and from entering Sj's market is* $\Theta_k\, M_{kj} - F$*. Pk sells more than Sj if Pk is a stronger seller than*

Sj *(i.e.,* $Q^P_{kj} > Q^S_{kj}$ *if* $\frac{\overline{\Delta}_{kj}}{\Delta_{kj}} > 1$*).*

Proposition 1. *Suppose platform entry is feasible (Condition FS holds). In the monopolistic platform setting, both sellers sell on P and P enters both sellers' product markets in equilibrium. Sj's equilibrium profit is* $\frac{\Theta}{\beta^S_j}\left[\frac{(2+\Omega_j)\,\Delta_{Pj}}{8+\Omega_j}\right]^2$*, and P's equilibrium profit is* $\Theta\, M_{P1} + \Theta\, M_{P2} - 2\,F$*.*

Lemma 7. *Suppose Condition FS holds and* $\frac{\Theta_1}{\Theta_2} \geq 1$*. Further suppose both platforms commit not to enter. Then Sj* $(j \in \{1,2\})$ *is indifferent between selling on P1 and selling on P2 if* $\frac{\Theta_1}{\Theta_2} = 1$*, whereas Sj sells on P1 if* $\frac{\Theta_1}{\Theta_2} > 1$*.*

Lemma 8. *Suppose Condition FS holds and* $\frac{\Theta_1}{\Theta_2} \geq 1$*. Further suppose platforms both make no commitment. If* $\frac{c^P_{2j}}{c^P_{1j}} < 1$*, then Sj will sell on P1. If* $\frac{c^P_{2j}}{c^P_{1j}} > 1$*, then Sj will: (i) sell on P1 when* $\frac{\Theta_1}{\Theta_2} > \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2$*; and (ii) sell on P2 when* $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2$*.*

Lemma 9. *Suppose Condition FS holds and* $\frac{\Theta_1}{\Theta_2} \geq 1$*. Further suppose P1 commits not to enter and P2 makes no commitment. Then Sj* $(j \in \{1,2\})$ *will sell on P1.*

Lemma 10. *Suppose Condition FS holds and* $\frac{\Theta_1}{\Theta_2} \geq 1$*. Further suppose P1 makes no commitment and P2 commits to no entry. Then Sj* $(j \in \{1,2\})$ *will: (i) sell on P1 if* $\frac{\Theta_1}{\Theta_2} > \phi_j$*; and (ii) sell on P2 if* $\frac{\Theta_1}{\Theta_2} < \phi_j$*.*

Proposition 2. *Suppose platform entry is feasible (Condition FS holds), a third-party seller benefits more from no competition than from reduced competition (Assumption BC holds), and* $\frac{\Theta_1}{\Theta_2} \geq 1$*. Then in equilibrium: (i) if* $\frac{\Theta_1}{\Theta_2} > \phi_2$*, P1 makes no commitment, and both S1 and S2 sell on P1; (ii) if* $\frac{\Theta_1}{\Theta_2} \in (\,\phi_1, \phi_2\,)$*, P1 makes no commitment whereas P2 commits not to enter, and S1 sells on P1 whereas S2 sells on P2; (iii) if* $\frac{\Theta_1}{\Theta_2} \in (\,1, \phi_1\,)$*, P1 commits not to enter, and both S1 and S2 sell on P1; and (iv) if* $\frac{\Theta_1}{\Theta_2} = 1$*, both platforms commit not to enter, and each seller is indifferent between selling on P1 and selling on P2.*

Condition FS $\Theta_k\, M_{k2} - \frac{\Theta_k\left[\widetilde{\Delta}_{k1}\right]^2}{8\, b^S_1} - \frac{\Theta_k\left[\widetilde{\Delta}_{k2}\right]^2}{8\, b^S_2} \ < \ F \ < \ \min\{\, \Theta_k\, M_{k2} - \frac{\Theta_k\left[\widetilde{\Delta}_{k2}\right]^2}{8\, b^S_2},\, \Theta_k\, M_{k1} - \frac{\Theta_k\left[\widetilde{\Delta}_{k1}\right]^2}{8\, b^S_1} - \frac{\Theta_k\left[\widetilde{\Delta}_{k2}\right]^2}{8\, b^S_2}\}$.

Assumption BC if $c^P_{1j} < c^P_{2j}$, then $\phi_1 > \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2$ for $j \in \{1,2\}$.

1 Multi-homing

In this section, sellers have the option to choose between single-homing, where they sell exclusively on one platform, and multi-homing, where they sell on both platforms.

I now consider a game in which platforms P1 and P2 initially decide whether to commit to not acting as sellers or to make no such commitment. Following these commitments, sellers choose to either sell exclusively on one platform (single-homing) or on both platforms (multi-homing). For those opting for single-homing, the decision of which platform to choose is also made. Platforms that have not made any commitments then decide whether to enter the market and, if so, which market to enter. Subsequently, P1 and P2 set their per-unit commissions. Finally, each active seller determines the profit-maximizing price for its products.

In this section, platforms launch exclusive programs that incentivize sellers to engage in single-homing by offering increased visibility for those who sell exclusively on one platform. Seller Sj ($j \in \{1,2\}$) receives an exogenous boost σB_k from platform Pk ($k \in \{1,2\}$), where $\sigma = 1$ if Sj sells on both platforms (multi-homing) and $\sigma > 1$ if Sj sells exclusively on Pk (single-homing). Therefore, σ measures the platform's rewards for single-homing sellers. Lemma 3 implies that Sj's profit from selling on Pk is $\frac{\sigma\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{16\, b_j^S}$, where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.

Assumption 1. *$\Theta_1 > \Theta_2$, $c_1^S < c_2^S$, $c_{k1}^P = c_{k2}^P$, and $c_{1j}^P > c_{2j}^P$ for $k, j \in \{1,2\}$.*

Assumption 1 pertains to the setting where P1 is a stronger platform (i.e., $\Theta_1 > \Theta_2$) but a weaker seller (i.e., $c_{1j}^P > c_{2j}^P$) than P2, S1 is a stronger seller than S2 (i.e., $c_1^S < c_2^S$, and therefore $\Delta_{k1} > \Delta_{k2}$ for $k \in \{1,2\}$), and the imitation cost of each seller is the same for Pk (i.e., $c_{k1}^P = c_{k2}^P$).

Lemma 11. *Suppose Sj faces no competition from Pk when Sj sells on Pk ($j, k \in \{1,2\}$). Then Sj's profit from selling on Pk is $\frac{\sigma\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{16\, b_j^S}$, and Pk's profit from the commission it collects from Sj is $\frac{\sigma\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{8\, b_j^S}$, where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.*

<u>Proof</u>**.** Since Sj receives an exogenous boost σB_k from platform Pk, where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk, Lemma 3 implies that in the absence of competition in the seller market, Sj's profit from selling on Pk is $\frac{\sigma\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{16\, b_j^S}$, and P$k$'s profit from the commission it collects from Sj is $\frac{\sigma\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{8\, b_j^S}$, where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk. ■

Lemma 12. *Suppose Sj competes against Pk when Sj sells on Pk ($j, k \in \{1,2\}$). Then Sj's profit from selling on Pk is $\frac{\sigma\Theta_k}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{kj}}{8+\Omega_j}\right]^2$, and Pk's profit from the commission it collects from Sj is $\sigma\Theta_k\, M_{kj} - F$, where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.*

Proof. Since Sj receives an exogenous boost σB_k from platform Pk, where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk, Lemma 6 implies that in the presence of competition in the seller market, Sj's profit from selling on Pk is $\frac{\sigma\Theta_k}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{kj}}{8+\Omega_j}\right]^2$, and P$k$'s profit from the commission it collects from Sj is $\sigma\Theta_k\, M_{kj} - F$, where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk. ■

Lemma 13. *Suppose Condition FS and Assumption 1 hold, and both platforms commit not to enter. Then Sj ($j \in \{1,2\}$) sells on both platforms if $\sigma < 1 + \frac{\Theta_2}{\Theta_1}$, whereas Sj sells on P1 if $\sigma > 1 + \frac{\Theta_2}{\Theta_1}$.*

Proof. Lemma 11 implies that Sj's profit is $\frac{\sigma\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{16\, b_j^S}$ if Sj sells exclusively on Pk. Sj's profit is $\frac{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\, b_j^S} + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\, b_j^S}$ if Sj sells on both platforms. Sj's profit is higher when Sj sells on P1 than when selling on P2 because

$$\frac{\sigma\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\, b_j^S} > \frac{\sigma\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\, b_j^S} \Leftrightarrow \Theta_1 > \Theta_2. \tag{14}$$

The last inequality in (14) reflects $\widetilde{\Delta}_{1j} = \widetilde{\Delta}_{2j}$ from (10). The last inequality in (14) holds due to $\Theta_1 > \Theta_2$ by assumption.

Sj's profit is higher when Sj sells on both platforms than when selling on P1 if:

$$\frac{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\, b_j^S} + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\, b_j^S} > \frac{\sigma\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\, b_j^S} \Leftrightarrow 1 + \frac{\Theta_2}{\Theta_1} > \sigma. \tag{15}$$

Sj's profit is higher when Sj sells on P1 than when selling on both platforms if:

$$\frac{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\, b_j^S} + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\, b_j^S} < \frac{\sigma\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\, b_j^S} \Leftrightarrow 1 + \frac{\Theta_2}{\Theta_1} < \sigma. \tag{16}$$

(15) and (16) reflect $\widetilde{\Delta}_{1j} = \widetilde{\Delta}_{2j}$ from (10). ■

Lemma 14. *Suppose Condition FS and Assumption 1 hold, and platforms both make no commitment. Then Sj ($j \in \{1,2\}$) sells on both platforms if $\sigma < 1 + \frac{\Theta_2[\Delta_{2j}]^2}{\Theta_1[\Delta_{1j}]^2}$. Sj sells on P1 if $\sigma > 1 + \frac{\Theta_2[\Delta_{2j}]^2}{\Theta_1[\Delta_{1j}]^2}$.*

Proof. Lemma 12 implies that Sj's profit is $\frac{\sigma\Theta_k}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{kj}}{8+\Omega_j}\right]^2$ if Sj sells exclusively on Pk. Sj's profit is $\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j}\right]^2$ if Sj sells on both platforms. Sj's profit is higher when Sj sells on P1 than when selling on P2 because

$$\frac{\sigma\Theta_1}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{1j}}{8+\Omega_j}\right]^2 > \frac{\sigma\Theta_2}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{2j}}{8+\Omega_j}\right]^2 \Leftrightarrow \Theta_1\left[\Delta_{1j}\right]^2 > \Theta_2\left[\Delta_{2j}\right]^2. \tag{17}$$

The last inequality in (17) holds because $\Theta_1 > \Theta_2$ by assumption and $\Delta_{1j} > \Delta_{2j}$ due to $c_{1j}^P > c_{2j}^P$ by assumption. Therefore, Sj's profit is higher when Sj sells on both platforms than when selling on P1 if:

$$\frac{\Theta_1}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{2j}}{8+\Omega_j}\right]^2 > \frac{\sigma\Theta_1}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{1j}}{8+\Omega_j}\right]^2$$

$$\Leftrightarrow \Theta_1\left[\Delta_{1j}\right]^2 + \Theta_2\left[\Delta_{2j}\right]^2 > \sigma\Theta_1\left[\Delta_{1j}\right]^2 \Leftrightarrow 1 + \frac{\Theta_2\left[\Delta_{2j}\right]^2}{\Theta_1\left[\Delta_{1j}\right]^2} > \sigma.$$

Sj's profit is higher when Sj sells on P1 than when selling on both platforms if:

$$\frac{\Theta_1}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{2j}}{8+\Omega_j}\right]^2 < \frac{\sigma\Theta_1}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{1j}}{8+\Omega_j}\right]^2$$

$$\Leftrightarrow \Theta_1\left[\Delta_{1j}\right]^2 + \Theta_2\left[\Delta_{2j}\right]^2 < \sigma\Theta_1\left[\Delta_{1j}\right]^2 \Leftrightarrow 1 + \frac{\Theta_2\left[\Delta_{2j}\right]^2}{\Theta_1\left[\Delta_{1j}\right]^2} < \sigma. \quad \blacksquare$$

Lemma 15. *Suppose Condition FS and Assumption 1 hold, P1 commits not to enter and P2 makes no commitment. Then Sj sells on both platforms if* $\sigma < 1 + \frac{16\Theta_2[1-\Omega_j][(2+\Omega_j)\Delta_{2j}]^2}{\Theta_1[8+\Omega_j]^2\left[\widetilde{\Delta}_{1j}\right]^2}$, *Sj sells on P1 if* $\sigma > 1 + \frac{16\Theta_2[1-\Omega_j][(2+\Omega_j)\Delta_{2j}]^2}{\Theta_1[8+\Omega_j]^2\left[\widetilde{\Delta}_{1j}\right]^2}$.

<u>Proof</u>**.** Condition FS ensures that P2 will enter Sj's market if Sj sells on P2 ($j \in \{1,2\}$). Lemma 11 implies that Sj's profit is $\frac{\sigma\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\,b_j^S}$ if Sj sells exclusively on Pif 1. Lemma 12 implies that Sj's profit is $\frac{\sigma\Theta_2}{\beta_j^S}\left[\frac{(2+\Omega_j)\Delta_{2j}}{8+\Omega_j}\right]^2$ if Sj sells exclusively on P2. (11) implies that:

$$\frac{\sigma\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\,b_j^S} > \frac{\sigma\Theta_2}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{2j}}{8+\Omega_j}\right]^2 \quad\Leftrightarrow\quad \frac{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\,\beta_j^S\,[\,1-\Omega_j\,]} > \frac{\Theta_2}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{2j}}{8+\Omega_j}\right]^2$$

$$\Leftrightarrow\quad \frac{\Theta_1}{\Theta_2} > \left[\frac{4\sqrt{1-\Omega_j}\,(\,2+\Omega_j\,)}{8+\Omega_j}\frac{\Delta_{2j}}{\widetilde{\Delta}_{1j}}\right]^2. \tag{18}$$

(18) holds because

$$\frac{\Theta_1}{\Theta_2} \;\geq\; 1 \;>\; \left[\frac{4\sqrt{1-\Omega_j}\,(\,2+\Omega_j\,)}{8+\Omega_j}\frac{\Delta_{2j}}{\widetilde{\Delta}_{1j}}\right]^2. \tag{19}$$

The last inequality in (19) holds because

$$\frac{4\sqrt{1-\Omega_j}\,[\,2+\Omega_j\,]}{8+\Omega_j} \;<\; 1 \;\;\text{and}\;\; \frac{\Delta_{2j}}{\widetilde{\Delta}_{1j}} \;<\; 1. \tag{20}$$

The first inequality in (20) holds because

$$\frac{\partial\left(\sqrt{1-\Omega_j}\frac{4[\,2+\Omega_j\,]}{8+\Omega_j}\right)}{\partial\Omega_j} \;=\; \sqrt{1-\Omega_j}\,\frac{\partial\left(\frac{4[\,2+\Omega_j\,]}{8+\Omega_j}\right)}{\partial\Omega_j} + \frac{\partial\left(\sqrt{1-\Omega_j}\right)}{\partial\Omega_j}\,\frac{4\,[\,2+\Omega_j\,]}{8+\Omega_j}$$

$$=\;\sqrt{1-\Omega_j}\,\frac{4\,[\,8+\Omega_j\,]-4\,[\,2+\Omega_j\,]}{[\,8+\Omega_j\,]^2} - \frac{1}{2}\,\frac{1}{\sqrt{1-\Omega_j}}\,\frac{4\,[\,2+\Omega_j\,]}{8+\Omega_j}$$

$$=\;\frac{24\sqrt{1-\Omega_j}}{[\,8+\Omega_j\,]^2} - \frac{2\,[\,2+\Omega_j\,]}{\sqrt{1-\Omega_j}\;[\,8+\Omega_j\,]} \;=\; \frac{2}{8+\Omega_j}\left[\,\frac{12\sqrt{1-\Omega_j}}{8+\Omega_j} - \frac{[\,2+\Omega_j\,]}{\sqrt{1-\Omega_j}}\,\right] \;<\; 0, \tag{21}$$

and therefore, for $\Omega_j \in (0,1)$,

$$\frac{4\sqrt{1-\Omega_j}\,[\,2+\Omega_j\,]}{8+\Omega_j} \;<\; \max\frac{4\sqrt{1-\Omega_j}\,[\,2+\Omega_j\,]}{8+\Omega_j} \;=\; \frac{4\sqrt{1-0}\,[\,2+0\,]}{8+0} \;=\; 1. \tag{22}$$

The last inequality in (21) holds because

$$\frac{12\sqrt{1-\Omega_j}}{8+\Omega_j} \;<\; \frac{[\,2+\Omega_j\,]}{\sqrt{1-\Omega_j}} \;\;\Leftrightarrow\;\; 12\,[\,1-\Omega_j\,] \;<\; [\,2+\Omega_j\,]\,[\,8+\Omega_j\,]$$

$$\Leftrightarrow\;\; 12-12\,\Omega_j \;<\; 16+[\Omega_j]^2+10\,\Omega_j \;\;\Leftrightarrow\;\; 4+[\Omega_j]^2+22\,\Omega_j \;>\; 0.$$

The last inequality in (20) holds because

$$\widetilde{\Delta}_{1j} \;=\; \widetilde{\Delta}_{2j} \;>\; \Delta_{2j}. \tag{23}$$

The equality in (23) reflects (10) and the inequality in (23) reflects (12).

(18)- (23) imply that Sj's profit is higher when it sells on P1 than when it sells on P2.

Therefore, Sj's profit is higher when Sj sells on both platforms than when selling on one platform if:

$$\frac{\Theta_1\left[\,\widetilde{\Delta}_{1j}\,\right]^2}{16\,b_j^S} + \frac{\Theta_2}{\beta_j^S}\left[\,\frac{(\,2+\Omega_j\,)\;\Delta_{2j}}{8+\Omega_j}\,\right]^2 > \frac{\sigma\Theta_1\left[\,\widetilde{\Delta}_{1j}\,\right]^2}{16\,b_j^S}$$

$$\Leftrightarrow \frac{\Theta_1\left[\,\widetilde{\Delta}_{1j}\,\right]^2}{16\,\beta_j^S\,[\,1-\Omega_j\,]} + \frac{\Theta_2}{\beta_j^S}\left[\,\frac{(\,2+\Omega_j\,)\;\Delta_{2j}}{8+\Omega_j}\,\right]^2 > \frac{\sigma\Theta_1\left[\,\widetilde{\Delta}_{1j}\,\right]^2}{16\,\beta_j^S\,[\,1-\Omega_j\,]} \tag{24}$$

$$\Leftrightarrow 1 + \frac{\Theta_2 \left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j} \right]^2}{\frac{\Theta_1 \left[\widetilde{\Delta}_{1j} \right]^2}{16[1-\Omega_j]}} > \sigma \Leftrightarrow \sigma < 1 + \frac{16\Theta_2\,[1-\Omega_j\,]\,[\,(2+\Omega_j\,)\,\Delta_{2j}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2 \left[\widetilde{\Delta}_{1j} \right]^2}.$$

(24) reflects (11).

Sj's profit is higher when Sj sells on P1 than when selling on both platforms if:

$$\frac{\Theta_1 \left[\widetilde{\Delta}_{1j} \right]^2}{16\, b_j^S} + \frac{\Theta_2}{\beta_j^S} \left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j} \right]^2 < \frac{\sigma\Theta_1 \left[\widetilde{\Delta}_{1j} \right]^2}{16\, b_j^S}$$

$$\Leftrightarrow \frac{\Theta_1 \left[\widetilde{\Delta}_{1j} \right]^2}{16\,\beta_j^S\,[1-\Omega_j\,]} + \frac{\Theta_2}{\beta_j^S} \left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j} \right]^2 < \frac{\sigma\Theta_1 \left[\widetilde{\Delta}_{1j} \right]^2}{16\,\beta_j^S\,[1-\Omega_j\,]}$$

$$\Leftrightarrow 1 + \frac{\Theta_2 \left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j} \right]^2}{\frac{\Theta_1 \left[\widetilde{\Delta}_{1j} \right]^2}{16[1-\Omega_j]}} < \sigma \Leftrightarrow \sigma > 1 + \frac{16\Theta_2\,[1-\Omega_j\,]\,[\,(2+\Omega_j\,)\,\Delta_{2j}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2 \left[\widetilde{\Delta}_{1j} \right]^2}. \quad \blacksquare$$

Lemma 16. *Suppose Condition FS and Assumption 1 hold, P2 commits not to enter and P1 makes no commitment. Then Sj sells on both platforms if: (i) either* $\frac{\Theta_1}{\Theta_2} > \phi_j$ *and* $\sigma < 1 + \frac{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}{16\Theta_1[1-\Omega_j][(2+\Omega_j)\,\Delta_{1j}]^2}$*; (ii) or* $\frac{\Theta_1}{\Theta_2} < \phi_j$ *and* $\sigma < 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\,\Delta_{1j}]^2}{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}$*, Sj sells on P1 if* $\frac{\Theta_1}{\Theta_2} > \phi_j$ *and* $\sigma > 1 + \frac{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}{16\Theta_1[1-\Omega_j][(2+\Omega_j)\,\Delta_{1j}]^2}$*, and Sj sells on P2 if* $\frac{\Theta_1}{\Theta_2} < \phi_j$ *and* $\sigma > 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\,\Delta_{1j}]^2}{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}$.

Proof. Condition FS ensures that P1 will enter Sj's market if Sj sells on P1 ($j \in \{1,2\}$). Lemma 11 implies that Sj's profit is $\frac{\sigma\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\, b_{2j}^S}$ if Sj sells exclusively on P2. Lemma 12 implies that Sj's profit is $\frac{\sigma\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2$ if Sj sells exclusively on P1. (11) and (4) imply that:

$$\frac{\sigma\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 \gtreqless \frac{\sigma\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\, b_{2j}^S} \quad \Leftrightarrow \quad \frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 \gtreqless \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,\beta_j^S\,[1-\Omega_j\,]}$$

$$\Leftrightarrow \quad \frac{\Theta_1}{\Theta_2} \gtreqless \left[\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,(2+\Omega_j)}\frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}}\right]^2 \quad \Leftrightarrow \quad \frac{\Theta_1}{\Theta_2} \gtreqless \phi_j. \tag{25}$$

(22) implies that $\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}[2+\Omega_j]} > 1$. (12) and (10) imply that $\widetilde{\Delta}_{2j} = \widetilde{\Delta}_{1j} > \Delta_{1j}$. Therefore,

$$\phi_j > 1. \tag{26}$$

Lemmas 11 and 12 imply Sj's profit is $\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\, b_j^S}$ if Sj sells on both

platforms.
<u>Case I</u>. $\frac{\Theta_1}{\Theta_2} > \phi_j$.

(25) implies that Sj's profit is higher when it sells on P1 than when it sells on P2. (11) implies that:

$$\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,b_j^S} \gtrless \frac{\sigma\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2$$

$$\Leftrightarrow \frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,\beta_j^S\,[1-\Omega_j]} \gtrless \frac{\sigma\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2$$

$$\Leftrightarrow 1 + \frac{\frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,\beta_j^S[1-\Omega_j]}}{\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\Delta_{1j}}{8+\Omega_j}\right]^2} \gtrless \sigma \Leftrightarrow \sigma \lessgtr 1 + \frac{\Theta_2\,[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}{16\Theta_1\,[1-\Omega_j]\,[(2+\Omega_j)\,\Delta_{1j}]^2}. \tag{27}$$

(27) implies that Sj's profit is higher when it sells on both platforms than when it sells on P1 if $\sigma < 1 + \frac{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{1j}]^2}$, and S$j$'s profit is higher when it sells on P1 than when it sells on both platforms if $\sigma > 1 + \frac{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{1j}]^2}$.
<u>Case II</u>. $\frac{\Theta_1}{\Theta_2} < \phi_j$.

(25) implies that Sj's profit is higher when it sells on P2 than when it sells on P1. (11) implies that:

$$\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,b_j^S} \gtrless \frac{\sigma\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,b_{2j}^S}$$

$$\Leftrightarrow \frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 + \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,\beta_j^S\,[1-\Omega_j]} \gtrless \frac{\sigma\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,\beta_j^S\,[1-\Omega_j]}$$

$$\Leftrightarrow \frac{\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\Delta_{1j}}{8+\Omega_j}\right]^2}{\frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{16\,\beta_j^S[1-\Omega_j]}} + 1 \gtrless \sigma \Leftrightarrow \sigma \lessgtr 1 + \frac{16\Theta_1\,[1-\Omega_j]\,[(2+\Omega_j)\,\Delta_{1j}]^2}{\Theta_2\,[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}. \tag{28}$$

(28) implies that Sj's profit is higher when it sells on both platforms than when it sells on P2 if $\sigma < 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{1j}]^2}{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}$, and S$j$'s profit is higher when it sells on P2 than when it sells on both platforms if $\sigma > 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{1j}]^2}{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{2j}\right]^2}$. ■

Lemmas 13, 14, 15, and 16 imply that each seller prefers single-homing if the platform's rewards for single-homing sellers (σ) exceed the benefits of multi-homing. The benefits of multi-homing are: (i) $1 + \frac{\Theta_2}{\Theta_1}$ if both platforms commit not to enter; (ii) $1 + \frac{\Theta_2[\Delta_{2j}]^2}{\Theta_1[\Delta_{1j}]^2}$ if neither platform commits; (iii) $1 + \frac{16\Theta_2[1-\Omega_j][(2+\Omega_j)\Delta_{2j}]^2}{\Theta_1[8+\Omega_j]^2[\widetilde{\Delta}_{1j}]^2}$ if P1 commits not to enter and P2 does not; and (iv) $1 + \frac{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{2j}]^2}{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{1j}]^2}$ if P1 is stronger and P2 commits not to enter while P1 does not; and (v) $1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{1j}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{2j}]^2}$ if P1 is relatively stronger and P2 commits not to enter while P1 does not. Conversely, sellers opt for multi-homing if the cumulative profits from both platforms exceed the additional rewards offered for single-homing.

Lemma 17. *Suppose* $\frac{\Theta_1}{\Theta_2} > \phi_{1j}$ *($j \in \{1,2\}$) and Assumption 1 holds. Then* $1 > \frac{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{2j}]^2}{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{1j}]^2} > \frac{\Theta_2}{\Theta_1} > \frac{\Theta_2[\Delta_{2j}]^2}{\Theta_1[\Delta_{1j}]^2} > \frac{16\Theta_2[1-\Omega_j][(2+\Omega_j)\Delta_{2j}]^2}{\Theta_1[8+\Omega_j]^2[\widetilde{\Delta}_{1j}]^2}$.

Proof. Assumption 1 implies $\Delta_{1j} > \Delta_{2j}$ because $\Delta_{kj} \equiv \alpha_j - \beta_j^S c_j^S + \eta_j c_{kj}^P$ by definition and $c_{1j}^P > c_{2j}^P$ from Assumption 1. Therefore, $\frac{[\Delta_{2j}]^2}{[\Delta_{1j}]^2} < 1$, and thus,

$$\frac{\Theta_2}{\Theta_1} > \frac{\Theta_2\,[\Delta_{2j}]^2}{\Theta_1\,[\Delta_{1j}]^2}. \tag{29}$$

(20) implies that:

$$\frac{[\,8+\Omega_j\,]^2}{16\,[\,1-\Omega_j\,]\,[\,2+\Omega_j\,]^2} > 1 \quad \text{and} \quad \frac{16\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\,]^2}{[\,8+\Omega_j\,]^2} < 1. \tag{30}$$

(30) implies that:

$$\frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\,\Delta_{2j}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\widetilde{\Delta}_{1j}\right]^2} < \frac{\Theta_2\,[\Delta_{2j}]^2}{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2} < \frac{\Theta_2\,[\Delta_{2j}]^2}{\Theta_1\,[\Delta_{1j}]^2};\ \text{and} \tag{31}$$

$$\frac{\Theta_2\,[\,8+\Omega_j\,]^2\left[\widetilde{\Delta}_{2j}\right]^2}{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\,\Delta_{1j}\,]^2} > \frac{\Theta_2\left[\widetilde{\Delta}_{2j}\right]^2}{\Theta_1\,[\Delta_{1j}]^2} > \frac{\Theta_2}{\Theta_1}. \tag{32}$$

The last inequality in (31) holds because $\widetilde{\Delta}_{1j} > \Delta_{1j}$ from (12). The last inequality in (32) holds because $\widetilde{\Delta}_{2j} = \widetilde{\Delta}_{1j} > \Delta_{1j}$ from (10) and (12).

(4) implies that when $\frac{\Theta_1}{\Theta_2} > \phi_{1j}$:

$$\frac{\Theta_1}{\Theta_2} > \left[\left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,(\,2+\Omega_j\,)}\right)\frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}}\right]^2$$

$$\Rightarrow \frac{\Theta_1}{\Theta_2}\frac{\Theta_2}{\Theta_1} > \left[\left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,(\,2+\Omega_j\,)}\right)\frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}}\right]^2\frac{\Theta_2}{\Theta_1}$$

$$\Rightarrow 1 > \frac{\Theta_2 \, [\, 8 + \Omega_j \,]^2 \left[\widetilde{\Delta}_{2j} \right]^2}{16 \Theta_1 \, [\, 1 - \Omega_j \,] \, [\, (\, 2 + \Omega_j \,) \; \Delta_{1j} \,]^2}. \tag{33}$$

(29), (31), (32) and (33) imply that when $\frac{\Theta_1}{\Theta_2} > \phi_{1j}$:

$$1 > \frac{\Theta_2 \, [\, 8 + \Omega_j \,]^2 \left[\widetilde{\Delta}_{2j} \right]^2}{16 \Theta_1 \, [\, 1 - \Omega_j \,] \, [\, (\, 2 + \Omega_j \,) \; \Delta_{1j} \,]^2} > \frac{\Theta_2}{\Theta_1} > \frac{\Theta_2 \, [\Delta_{2j}]^2}{\Theta_1 \, [\Delta_{1j}]^2} > \frac{16 \Theta_2 \, [\, 1 - \Omega_j \,] \, [\, (\, 2 + \Omega_j \,) \; \Delta_{2j} \,]^2}{\Theta_1 \, [\, 8 + \Omega_j \,]^2 \left[\widetilde{\Delta}_{1j} \right]^2}. \; \blacksquare \tag{34}$$

Lemma 18. *Suppose* $\frac{\Theta_1}{\Theta_2} \in \left(\sqrt{\phi_{1j}}, \phi_{1j} \right)$ *and Assumption 1 holds. Further suppose* $\Omega_1 = \Omega_2 \in (0.5, 1)$*. Then* $\frac{\varsigma_{11} + \varsigma_{12}}{\frac{1}{2 b_1^S} \left[\frac{\widetilde{\Delta}_{11}}{4} \right]^2 + \frac{1}{2 b_2^S} \left[\frac{\widetilde{\Delta}_{12}}{4} \right]^2} - 1 > 1 > \frac{16 \Theta_1 [\, 1 - \Omega_j \,] [\, (\, 2 + \Omega_j \,) \, \Delta_{11} \,]^2}{\Theta_2 [\, 8 + \Omega_j \,]^2 \left[\widetilde{\Delta}_{21} \right]^2} > \frac{16 \Theta_1 [\, 1 - \Omega_j \,] [\, (\, 2 + \Omega_j \,) \, \Delta_{12} \,]^2}{\Theta_2 [\, 8 + \Omega_j \,]^2 \left[\widetilde{\Delta}_{22} \right]^2} > \frac{\Theta_2}{\Theta_1} > \frac{\Theta_2 [\Delta_{21}]^2}{\Theta_1 [\Delta_{11}]^2} > \frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2} > \frac{16 \Theta_2 [\, 1 - \Omega_j \,] [\, (\, 2 + \Omega_j \,) \, \Delta_{21} \,]^2}{\Theta_1 [\, 8 + \Omega_j \,]^2 \left[\widetilde{\Delta}_{11} \right]^2} > \frac{16 \Theta_2 [\, 1 - \Omega_j \,] [\, (\, 2 + \Omega_j \,) \, \Delta_{22} \,]^2}{\Theta_1 [\, 8 + \Omega_j \,]^2 \left[\widetilde{\Delta}_{12} \right]^2}$.

Proof. (11), (12) and (8) imply that:

$$\begin{aligned}
\varsigma_{kj} \; > \; \frac{1}{b_j^S} \left[\frac{\widetilde{\Delta}_{kj}}{4} \right]^2 \quad &\Leftrightarrow \quad \varsigma_{kj} \; > \; \frac{1}{\beta_j^S \, [\, 1 - \Omega_j \,]} \left[\frac{\widetilde{\Delta}_{kj}}{4} \right]^2 \\
\Leftrightarrow \quad \frac{1}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} \Bigg\{ & \frac{\eta_j \, [\, 2 + \Omega_j \,] \; \Delta_{kj}}{8 + \Omega_j} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \; \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right] \\
& + \frac{\beta_j^S}{2} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \; \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right]^2 + \frac{\beta_j^P}{2} \left[\frac{(\, 2 + \Omega_j \,) \; \Delta_{kj}}{8 + \Omega_j} \right]^2 \Bigg\} \\
> \; \frac{1}{16 \, \beta_j^S \, [\, 1 - \Omega_j \,]} & \left[\Delta_{kj} + \frac{\eta_j}{\beta_j^P} \, \overline{\Delta}_{kj} \right]^2
\end{aligned} \tag{35}$$

$$\begin{aligned}
\Leftrightarrow \quad & \frac{\eta_j \, [\, 2 + \Omega_j \,] \; \Delta_{kj}}{8 + \Omega_j} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \; \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right] + \frac{\beta_j^S}{2} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \; \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right]^2 \\
& + \frac{\beta_j^P}{2} \left[\frac{(\, 2 + \Omega_j \,) \; \Delta_{kj}}{8 + \Omega_j} \right]^2 \; > \; \frac{\beta_j^P}{16} \left[\Delta_{kj} + \frac{\eta_j}{\beta_j^P} \, \overline{\Delta}_{kj} \right]^2 \\
\Leftrightarrow \quad & \frac{\eta_j \, [\, 2 + \Omega_j \,] \; \Delta_{kj} \, \overline{\Delta}_{kj}}{2 \, [\, 8 + \Omega_j \,]} + \frac{[\eta_j]^2 \, [\, 2 + \Omega_j \,]^2 \; [\Delta_{kj}]^2}{2 \, \beta_j^S \, [\, 8 + \Omega_j \,]^2} + \frac{\beta_j^P \, [\, 2 + \Omega_j \,]^2 \; [\Delta_{kj}]^2}{2 \, [\, 8 + \Omega_j \,]^2} \\
& + \frac{\beta_j^S}{8} \left[\overline{\Delta}_{kj} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \; \Delta_{kj}}{\beta_j^S \, (\, 8 + \Omega_j \,)} \right]^2 \; > \; \frac{\beta_j^P}{16} \left[\Delta_{kj} + \frac{\eta_j}{\beta_j^P} \, \overline{\Delta}_{kj} \right]^2
\end{aligned}$$

$$\Leftrightarrow \quad \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2}$$

$$+\;\frac{\beta_j^S}{4}\left[\left(\overline{\Delta}_{kj}\right)^2+\frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)^2\;(\Delta_{kj})^2}{\left(\beta_j^S\right)^2(\,8+\Omega_j\,)^2}+\frac{2\,\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right]\;>\;\frac{\beta_j^P}{8}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2}+\;\frac{\beta_j^S\;\left[\overline{\Delta}_{kj}\right]^2}{4}$$

$$+\;\frac{\beta_j^S\;\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{4\left[\beta_j^S\right]^2[\,8+\Omega_j\,]^2}+\frac{2\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}\;>\;\frac{\beta_j^P}{8}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{4\,\beta_j^S\;\left[\eta_j\right]^2[\,2+\Omega_j\,]^2+4\left[\beta_j^S\right]^2\beta_j^P\,[\,2+\Omega_j\,]^2+\beta_j^S\;\left[\eta_j\right]^2[\,2+\Omega_j\,]^2}{4\left[\beta_j^S\right]^2[\,8+\Omega_j\,]^2}\,[\Delta_{kj}]^2$$

$$+\;\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}+\frac{\beta_j^S\;\left[\overline{\Delta}_{kj}\right]^2}{4}\;>\;\frac{\beta_j^P}{8}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{4\left[\eta_j\right]^2+4\,\beta_j^S\,\beta_j^P+\left[\eta_j\right]^2}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2$$

$$+\;\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}+\frac{\beta_j^S\;\left[\overline{\Delta}_{kj}\right]^2}{4}\;>\;\frac{\beta_j^P}{8}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{\left[5\left(\eta_j\right)^2+4\,\beta_j^S\,\beta_j^P\right][\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]}+\beta_j^S\;\left[\overline{\Delta}_{kj}\right]^2$$

$$>\;\frac{\beta_j^P}{2}\left[[\Delta_{kj}]^2+\frac{\left(\eta_j\right)^2\left(\overline{\Delta}_{kj}\right)^2}{\left(\beta_j^P\right)^2}+\frac{2\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^P}\right]$$

$$\Leftrightarrow \quad \frac{\beta_j^S\,\beta_j^P\,[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{6\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\beta_j^S\;\left[\overline{\Delta}_{kj}\right]^2$$

$$>\;\frac{\beta_j^P\,[\Delta_{kj}]^2}{2}+\frac{\left[\eta_j\right]^2\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P}+\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$\Leftrightarrow \quad \frac{\beta_j^P\,[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2}-\frac{\beta_j^P\,[\Delta_{kj}]^2}{2}+\frac{6\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}-\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$+\;\beta_j^S\;\left[\overline{\Delta}_{kj}\right]^2-\frac{\left[\eta_j\right]^2\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P}\;>\;0$$

$$\Leftrightarrow \quad \frac{2\,[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2-[\,8+\Omega_j\,]^2}{2\,[\,8+\Omega_j\,]^2}\beta_j^P\;[\Delta_{kj}]^2+\frac{6\,[\,2+\Omega_j\,]-[\,8+\Omega_j\,]}{[\,8+\Omega_j\,]}\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$+ \frac{2\,\beta_j^P\,\beta_j^S - [\eta_j]^2}{2\,\beta_j^P} \left[\overline{\Delta}_{kj}\right]^2 \;>\; 0$$

$$\Leftrightarrow \quad \frac{2\,[\,5\,\Omega_j + 4\,]\,[\,2 + \Omega_j\,]^2 - [\,8 + \Omega_j\,]^2}{2\,[\,8 + \Omega_j\,]^2} \beta_j^P\,[\Delta_{kj}]^2 + \frac{[\,4 + 5\,\Omega_j\,]\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{[\,8 + \Omega_j\,]}$$

$$+ \frac{[\,2 - \Omega_j\,]\,\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2}{2} \;>\; 0. \tag{36}$$

(36) holds because $\Omega_j \in (0,1)$ and

$$2\,[\,5\,\Omega_j + 4\,]\,[\,2 + \Omega_j\,]^2 - [\,8 + \Omega_j\,]^2 \;>\; 0 \quad \Leftrightarrow \quad 2\,[\,5\,\Omega_j + 4\,]\,[\,2 + \Omega_j\,]^2 \;>\; [\,8 + \Omega_j\,]^2$$

$$\Leftrightarrow \quad 2\,[\,5\,\Omega_j + 4\,]\left[\,4 + (\Omega_j)^2 + 4\,\Omega_j\,\right] \;>\; 64 + [\Omega_j]^2 + 16\,\Omega_j$$

$$\Leftrightarrow \quad [\,5\,\Omega_j + 4\,]\left[\,8 + 2\,(\Omega_j)^2 + 8\,\Omega_j\,\right] \;>\; 64 + [\Omega_j]^2 + 16\,\Omega_j$$

$$\Leftrightarrow \quad 5\,\Omega_j\left[\,8 + 2\,(\Omega_j)^2 + 8\,\Omega_j\,\right] + 4\left[\,8 + 2\,(\Omega_j)^2 + 8\,\Omega_j\,\right] \;>\; 64 + [\Omega_j]^2 + 16\,\Omega_j$$

$$\Leftrightarrow \quad 40\,\Omega_j + 10\,[\Omega_j]^3 + 40\,[\Omega_j]^2 + 32 + 8\,[\Omega_j]^2 + 32\,\Omega_j - 64 - [\Omega_j]^2 - 16\,\Omega_j \;>\; 0$$

$$\Leftrightarrow \quad 56\,\Omega_j + 10\,[\Omega_j]^3 + 47\,[\Omega_j]^2 - 32 \;>\; 0 \quad \Leftrightarrow \quad \Omega_j\left[56\; + 10\,(\Omega_j)^2 + 47\Omega_j\right] \;>\; 32. \tag{37}$$

(37) holds because $\Omega_j \in (0.5, 1)$.
(35) implies that:

$$\varsigma_{11} + \varsigma_{12} > \frac{1}{b_1^S}\left[\frac{\widetilde{\Delta}_{11}}{4}\right]^2 + \frac{1}{b_2^S}\left[\frac{\widetilde{\Delta}_{12}}{4}\right]^2$$

$$\Leftrightarrow \varsigma_{11} + \varsigma_{12} - \frac{1}{2b_1^S}\left[\frac{\widetilde{\Delta}_{11}}{4}\right]^2 - \frac{1}{2b_2^S}\left[\frac{\widetilde{\Delta}_{12}}{4}\right]^2 > \frac{1}{2b_1^S}\left[\frac{\widetilde{\Delta}_{11}}{4}\right]^2 + \frac{1}{2b_2^S}\left[\frac{\widetilde{\Delta}_{12}}{4}\right]^2$$

$$\Leftrightarrow \frac{\varsigma_{11} + \varsigma_{12}}{\frac{1}{2b_1^S}\left[\frac{\widetilde{\Delta}_{11}}{4}\right]^2 + \frac{1}{2b_2^S}\left[\frac{\widetilde{\Delta}_{12}}{4}\right]^2} - 1 > 1. \tag{38}$$

$\frac{\Theta_1}{\Theta_2} \in \left(\sqrt{\phi_{1j}}, \phi_{1j}\right)$ and (38) imply that:

$$\frac{\Theta_1}{\Theta_2} < \phi_{11} < \left[\frac{\varsigma_{11} + \varsigma_{12}}{\frac{1}{2b_1^S}\left(\frac{\widetilde{\Delta}_{11}}{4}\right)^2 + \frac{1}{2b_2^S}\left(\frac{\widetilde{\Delta}_{12}}{4}\right)^2} - 1\right]\phi_{11}. \tag{39}$$

(4) and (39) imply that:

$$\frac{\Theta_1}{\Theta_2} < \left[\frac{\varsigma_{11} + \varsigma_{12}}{\frac{1}{2b_1^S}\left(\frac{\widetilde{\Delta}_{11}}{4}\right)^2 + \frac{1}{2b_2^S}\left(\frac{\widetilde{\Delta}_{12}}{4}\right)^2} - 1\right] \frac{[\,8 + \Omega_j\,]^2\left[\,\widetilde{\Delta}_{21}\,\right]^2}{16\,[\,1 - \Omega_j\,]\,[\,(\,2 + \Omega_j\,)\;\Delta_{11}\,]^2}$$

$$\Leftrightarrow \frac{16\Theta_1 \left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{11} \right]^2}{\Theta_2 \left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{21} \right]^2} < \frac{\varsigma_{11}+\varsigma_{12}}{\frac{1}{2b_1^S}\left(\frac{\widetilde{\Delta}_{11}}{4}\right)^2 + \frac{1}{2b_2^S}\left(\frac{\widetilde{\Delta}_{12}}{4}\right)^2} - 1 \tag{40}$$

(4) implies that when $\frac{\Theta_1}{\Theta_2} < \phi_{1j}$:

$$\frac{\Theta_1}{\Theta_2} < \left[\left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\left(2+\Omega_j \right)} \right) \frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}} \right]^2 \tag{41}$$

$$\Rightarrow \frac{\Theta_1}{\Theta_2} \frac{16\left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}{\left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2} < \left[\left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\left(2+\Omega_j \right)} \right) \frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}} \right]^2 \frac{16\left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}{\left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2}$$

$$\Rightarrow \frac{16\Theta_1 \left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}{\Theta_2 \left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2} < 1. \tag{42}$$

(41) implies that:

$$\frac{\Theta_1}{\Theta_2}\frac{\Theta_2}{\Theta_1} < \left[\left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\left(2+\Omega_j \right)} \right) \frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}} \right]^2 \frac{\Theta_2}{\Theta_1}$$

$$\Rightarrow 1 < \frac{\Theta_2 \left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2}{16\Theta_1 \left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}. \tag{43}$$

(42) and (43) imply that:

$$\frac{16\Theta_1 \left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}{\Theta_2 \left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2} < 1 < \frac{\Theta_2 \left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2}{16\Theta_1 \left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}. \tag{44}$$

(4), (30), $\frac{\Theta_1}{\Theta_2} \in \left(\sqrt{\phi_{1j}}, \phi_{1j} \right)$, and $\widetilde{\Delta}_{2j} > \Delta_{1j}$ from (13) imply that:

$$\frac{\Theta_1}{\Theta_2} > \left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\left(2+\Omega_j \right)} \right) \frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}} > 1 \Rightarrow \left[\frac{\Theta_1}{\Theta_2} \right]^2 > \frac{\left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2}{16\left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}$$

$$\Rightarrow \frac{16\Theta_1 \left[1-\Omega_j \right] \left[\left(2+\Omega_j \right) \Delta_{1j} \right]^2}{\Theta_2 \left[8+\Omega_j \right]^2 \left[\widetilde{\Delta}_{2j} \right]^2} > \frac{\Theta_2}{\Theta_1}. \tag{45}$$

(29), (31), (44), and (45) imply that:

$$1 > \frac{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{1j}\,]^2}{\Theta_2\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{2j}\,\right]^2} > \frac{\Theta_2}{\Theta_1} > \frac{\Theta_2\,[\Delta_{2j}]^2}{\Theta_1\,[\Delta_{1j}]^2} > \frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{2j}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{1j}\,\right]^2}. \tag{46}$$

Observe that:

$$\frac{\partial\left(\frac{\Delta_{2j}}{\Delta_{1j}}\right)}{\partial c_j^S} \stackrel{s}{=} \frac{\partial \Delta_{2j}}{\partial c_j^S}\,\Delta_{1j} - \frac{\partial \Delta_{1j}}{\partial c_j^S}\,\Delta_{2j} \;=\; -\beta_j^S\,\Delta_{1j} + \beta_j^S\,\Delta_{2j} \;=\; \beta_j^S\,[\,\Delta_{2j}-\Delta_{1j}\,] \;=\; \beta_j^S\,\eta_j\,[\,c_{2j}^P - c_{1j}^P\,] \;<\; 0. \tag{47}$$

The inequality in (47) holds because $c_{1j}^P > c_{2j}^P$ by assumption.

$c_1^S < c_2^S$ from Assumption 1 and (47) imply that:

$$\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 \;<\; \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2. \tag{48}$$

Further observe that:

$$\frac{\partial\left(\frac{\Delta_{1j}}{\widetilde{\Delta}_{2j}}\right)}{\partial c_j^S} \stackrel{s}{=} \frac{\partial \Delta_{1j}}{\partial c_j^S}\,\widetilde{\Delta}_{2j} - \frac{\partial \widetilde{\Delta}_{2j}}{\partial c_j^S}\,\Delta_{1j} \;=\; -\,\beta_j^S\,\widetilde{\Delta}_{2j} + b_j^S\,\Delta_{1j} \;<\; 0, \text{ and} \tag{49}$$

$$\frac{\partial\left(\frac{\Delta_{2j}}{\widetilde{\Delta}_{1j}}\right)}{\partial c_j^S} \stackrel{s}{=} \frac{\partial \Delta_{2j}}{\partial c_j^S}\,\widetilde{\Delta}_{1j} - \frac{\partial \widetilde{\Delta}_{1j}}{\partial c_j^S}\,\Delta_{2j} \;=\; -\,\beta_j^S\,\widetilde{\Delta}_{1j} + b_j^S\,\Delta_{2j} \;<\; 0. \tag{50}$$

The inequalities in (49) and (50) hold because $\beta_j^S > b_j^S$ from (11), $\widetilde{\Delta}_{2j} > \Delta_{1j}$ and $\widetilde{\Delta}_{1j} > \Delta_{2j}$ from (13).

$c_1^S < c_2^S$ from Assumption 1, (49), and (50) imply that:

$$\left[\frac{\Delta_{12}}{\widetilde{\Delta}_{22}}\right]^2 \;<\; \left[\frac{\Delta_{11}}{\widetilde{\Delta}_{21}}\right]^2 \text{ and } \left[\frac{\Delta_{22}}{\widetilde{\Delta}_{12}}\right]^2 \;<\; \left[\frac{\Delta_{21}}{\widetilde{\Delta}_{11}}\right]^2. \tag{51}$$

(46), (48), and (51) imply that:

$$1 > \frac{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{11}\,]^2}{\Theta_2\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{21}\,\right]^2} > \frac{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{12}\,]^2}{\Theta_2\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{22}\,\right]^2} > \frac{\Theta_2}{\Theta_1} > \frac{\Theta_2\,[\Delta_{21}]^2}{\Theta_1\,[\Delta_{11}]^2}$$

$$> \frac{\Theta_2\,[\Delta_{22}]^2}{\Theta_1\,[\Delta_{12}]^2} > \frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{21}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{11}\,\right]^2} > \frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{22}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{12}\,\right]^2}. \quad \blacksquare$$

Condition FR $\Theta_k\,M_{k2} - \frac{\Theta_k\left[\,\widetilde{\Delta}_{k1}\,\right]^2}{8\,b_1^S} - \frac{\Theta_k\left[\,\widetilde{\Delta}_{k2}\,\right]^2}{8\,b_2^S} \;<\; F \;<\; \min\{\,\Theta_k\,M_{k2} - \frac{\Theta_k\left[\,\widetilde{\Delta}_{k2}\,\right]^2}{8\,b_2^S},\, \Theta_k\,M_{k1} - \frac{\sigma\Theta_k\left[\,\widetilde{\Delta}_{k1}\,\right]^2}{8\,b_1^S} - \frac{\sigma\Theta_k\left[\,\widetilde{\Delta}_{k2}\,\right]^2}{8\,b_2^S}\}.$

Condition FR ensures that the feasibility of platform entry is maintained in scenarios allowing multi-homing.

Proposition 3. *Suppose* $\frac{\Theta_1}{\Theta_2} \in \left(\sqrt{\phi_{1j}}, \phi_{1j}\right)$ *and Assumption 1 holds. Further suppose* $\Omega_1 = \Omega_2$ *and Condition FR holds. Then in equilibrium: (i) if* $\sigma > 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{11}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{21}]^2}$*, both P1 and P2 commit not to enter, and both sellers sell on P1; (ii) if* $\sigma \in (1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{22}]^2},$ $1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{11}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{21}]^2})$*, P1 makes no commitment and P2 commits not to enter, and S1 sells on both platforms while S2 sells exclusively on P2; (iii) if* $\sigma \in (1 + \frac{\Theta_2[\Delta_{21}]^2}{\Theta_1[\Delta_{11}]^2},$ $1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{22}]^2})$*, P1 makes no commitment and P2 commits not to enter, and both sellers sell on both platforms; (iv) if* $\sigma \in \left(1 + \frac{\Theta_2[\Delta_{22}]^2}{\Theta_1[\Delta_{12}]^2}, 1 + \frac{\Theta_2[\Delta_{21}]^2}{\Theta_1[\Delta_{11}]^2}\right)$*, both platforms make no commitments, and S1 sells on both platforms and S2 sells on P1; and (v) if* $\sigma \in \left(1, 1 + \frac{\Theta_2[\Delta_{22}]^2}{\Theta_1[\Delta_{12}]^2}\right)$*, both platforms make no commitments, and each seller sells on both platforms.*

<u>Proof</u>**.** Condition FS ensures that each platform enters each seller's market if the platform makes no commitment. Since S1 and S2 sell independent products, S1's choice of platform is independent of S2's choice of platform.

<u>Case I</u>**.** $\sigma > 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{11}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{21}]^2}$. Lemma 18 implies that:

$$\sigma > 1+\frac{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\,\Delta_{11}\,]^2}{\Theta_2\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{21}\,\right]^2} > 1+\frac{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\,\Delta_{12}\,]^2}{\Theta_2\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{22}\,\right]^2} > 1+\frac{\Theta_2}{\Theta_1} > 1+\frac{\Theta_2\,[\Delta_{21}]^2}{\Theta_1\,[\Delta_{11}]^2}$$

$$> 1+\frac{\Theta_2\,[\Delta_{22}]^2}{\Theta_1\,[\Delta_{12}]^2} > 1+\frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\,\Delta_{21}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{11}\,\right]^2} > 1+\frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\,\Delta_{22}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{12}\,\right]^2}. \tag{52}$$

If P2 makes no commitment, (52), Lemmas 14 and 15 imply that both sellers sell on P1. Lemmas 11 and 12 imply that: (i) P1's profit is $\sigma\Theta_1 M_{11} - F + \sigma\Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\sigma\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\sigma\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1[\widetilde{\Delta}_{1j}]^2}{8\,b_j^S}$, i.e., P1 secures more profit by making no commitment than by committing to no entry. Therefore, if P2 makes no commitment, then P1 makes no commitment, and both S1 and S2 sell on P1 if $\sigma > 2$.

If P2 commits not to enter, (52), Lemmas 13 and 16 imply that both sellers: (i) sell on P2 if P1 makes no commitment; and (ii) sell on P1 if P1 commits not to enter. Lemmas 11 and 12 imply that P1's profit is: (i) $\frac{\sigma\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\sigma\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$ if P1 commits not to enter; and (ii) zero if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case. Consequently, if P2 commits not to enter, then P1 commits not to enter, and both S1 and S2 sell on P1 if $\sigma > 2$.

If P1 commits not to enter, (52), Lemmas 13 and 15 imply that both sellers sell on P1, regardless of P2's commitment. Therefore, if P1 commits not to enter, then P2 is indifferent between making no commitment and committing not to enter if $\sigma > 2$.

If P1 makes no commitment, (52), Lemmas 14 and 16 imply that both sellers: (i) sell on P1 if P2 makes no commitment; and (ii) sell on P2 if P2 commits not to enter. Lemmas 11 and 12 imply that P2's profit is: (i) $\frac{\sigma\Theta_2\left[\widetilde{\Delta}_{21}\right]^2}{8\,b_1^S} + \frac{\sigma\Theta_2\left[\widetilde{\Delta}_{22}\right]^2}{8\,b_2^S}$ if P2 commits not to enter; and (ii) zero if P2 makes no commitment. Therefore, P2 secures more profit by committing not to enter than by making no commitment in this case. Consequently, if P1 makes no commitment, then P2 commits not to enter, and both S1 and S2 sell on P1 if $\sigma > 2$.

Consequently, in equilibrium, both P1 and P2 commit not to enter, and both sellers sell on P1 if $\sigma > 2$.

<u>Case II</u>**.** $\sigma \in \left(1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{22}\right]^2}, 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{11}]^2}{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{21}\right]^2}\right)$. Lemma 18 implies that:

$$1+\frac{16\Theta_1\left[1-\Omega_1\right]\left[\left(2+\Omega_1\right)\Delta_{11}\right]^2}{\Theta_2\left[8+\Omega_1\right]^2\left[\widetilde{\Delta}_{21}\right]^2} > \sigma > 1+\frac{16\Theta_1\left[1-\Omega_j\right]\left[\left(2+\Omega_j\right)\Delta_{12}\right]^2}{\Theta_2\left[8+\Omega_j\right]^2\left[\widetilde{\Delta}_{22}\right]^2} > 1+\frac{\Theta_2}{\Theta_1} > 1+\frac{\Theta_2\left[\Delta_{21}\right]^2}{\Theta_1\left[\Delta_{11}\right]^2}$$

$$> 1+\frac{\Theta_2\left[\Delta_{22}\right]^2}{\Theta_1\left[\Delta_{12}\right]^2} > 1+\frac{16\Theta_2\left[1-\Omega_j\right]\left[\left(2+\Omega_j\right)\Delta_{21}\right]^2}{\Theta_1\left[8+\Omega_j\right]^2\left[\widetilde{\Delta}_{11}\right]^2} > 1+\frac{16\Theta_2\left[1-\Omega_j\right]\left[\left(2+\Omega_j\right)\Delta_{22}\right]^2}{\Theta_1\left[8+\Omega_j\right]^2\left[\widetilde{\Delta}_{12}\right]^2}. \tag{53}$$

If P2 makes no commitment, (53), Lemmas 14 and 15 imply that both sellers sell on P1. Lemmas 11 and 12 imply that: (i) P1's profit is $\sigma\Theta_1 M_{11} - F + \sigma\Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\sigma\Theta_1\left[\widetilde{\Delta}_{11}\right]^2}{8\,b_1^S} + \frac{\sigma\Theta_1\left[\widetilde{\Delta}_{12}\right]^2}{8\,b_2^S}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{8\,b_j^S}$, i.e., P1 secures more profit by making no commitment than by committing to no entry. Therefore, if P2 makes no commitment, then P1 makes no commitment, and both S1 and S2 sell on P1 in this case.

If P2 commits not to enter, (53), Lemmas 13 and 16 imply that: (i) both sellers sell on P1 if P1 commits not to enter; and (ii) S1 sells on both platforms and S2 sells on P2 if P1 makes no commitment. Lemmas 11 and 12 imply that P1's profit is: (i) $\Theta_1 M_{11} - F$ if P1 makes no commitment; and (ii) $\frac{\sigma\Theta_1\left[\widetilde{\Delta}_{11}\right]^2}{8\,b_1^S} + \frac{\sigma\Theta_1\left[\widetilde{\Delta}_{12}\right]^2}{8\,b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{11} - F > \frac{\sigma\Theta_1\left[\widetilde{\Delta}_{11}\right]^2}{8\,b_1^S} + \frac{\sigma\Theta_1\left[\widetilde{\Delta}_{12}\right]^2}{8\,b_2^S}$. Therefore, if P2 commits not to enter, then P1 makes no commitment in this case.

If P1 makes no commitment, (53), Lemmas 14 and 16 imply that: (i) both sellers sell on P1 if P2 makes no commitment; and (ii) S1 sells on both platforms and S2 sells on P2 if P2 commits not to enter. Lemmas 11 and 12 imply that P2's profit is: (i) zero if P2 makes no commitment; and (ii) $\frac{\Theta_2[\widetilde{\Delta}_{21}]^2}{8\,b_1^S} + \frac{\sigma\Theta_2[\widetilde{\Delta}_{22}]^2}{8\,b_2^S}$ if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment. Consequently, if P1 makes no commitment, then P2 commits not to enter, and S1 sells on both platforms while S2 sells exclusively on P2 in this case.

If P1 commits not to enter, (53), Lemmas 13 and 15 imply that both sellers sell on P1 regardless of P2's commitment.

Consequently, in equilibrium, P1 makes no commitment and P2 commits not to enter, and S1 sells on both platforms while S2 sells exclusively on P2 if $\sigma \in (1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{22}]^2}$, $1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{11}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{21}]^2})$.

<u>Case III</u>. $\sigma \in \left(1 + \frac{\Theta_2}{\Theta_1}, 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{22}]^2}\right)$. Lemma 18 implies that:

$$1+\frac{16\Theta_1\,[\,1-\Omega_1\,]\,[\,(\,2+\Omega_1\,)\;\Delta_{11}\,]^2}{\Theta_2\,[\,8+\Omega_1\,]^2\left[\,\widetilde{\Delta}_{21}\,\right]^2} > 1+\frac{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{12}\,]^2}{\Theta_2\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{22}\,\right]^2} > \sigma > 1+\frac{\Theta_2}{\Theta_1} > 1+\frac{\Theta_2\,[\Delta_{21}]^2}{\Theta_1\,[\Delta_{11}]^2}$$
$$> 1+\frac{\Theta_2\,[\Delta_{22}]^2}{\Theta_1\,[\Delta_{12}]^2} > 1+\frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{21}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{11}\,\right]^2} > 1+\frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{22}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{12}\,\right]^2}. \tag{54}$$

If P2 makes no commitment, (54), Lemmas 14 and 15 imply that both sellers sell on P1. Lemmas 11 and 12 imply that: (i) P1's profit is $\sigma\Theta_1\,M_{11} - F + \sigma\Theta_1\,M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\sigma\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\sigma\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1\,M_{1j} - F > \frac{\Theta_1[\widetilde{\Delta}_{1j}]^2}{8\,b_j^S}$, i.e., P1 secures more profit by making no commitment than by committing to no entry. Therefore, if P2 makes no commitment, then P1 makes no commitment, and both S1 and S2 sell on P1 in this case.

If P2 commits not to enter, (54), Lemmas 13 and 16 imply that: (i) both sellers sell on P1 if P1 commits not to enter; and (ii) both sellers sell on both platforms if P1 makes no commitment. Lemmas 11 and 12 imply that P1's profit is: (i) $\Theta_1\,M_{11} - F + \Theta_1\,M_{12} - F$ if P1 makes no commitment; and (ii) $\frac{\sigma\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\sigma\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1\,M_{11} - F > \frac{\sigma\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\sigma\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$, which implies that $\Theta_1\,M_{11} - F + \Theta_1\,M_{12} - F > \frac{\sigma\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\sigma\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$. Therefore, if P2 commits not to enter, then P1 makes no commitment.

If P1 makes no commitment, (54), Lemmas 14 and 16 imply that both sellers: (i) sell on P1 if P2 makes no commitment; and (ii) sell on both platforms if P2 commits not to enter. Lemmas 11 and 12 imply that P2's profit is: (i) zero if P2 makes no commitment; and (ii) $\frac{\Theta_2[\widetilde{\Delta}_{21}]^2}{8\,b_1^S} + \frac{\Theta_2[\widetilde{\Delta}_{22}]^2}{8\,b_2^S}$ if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment. Consequently, if P1 makes no commitment, then P2 commits not to enter, and both sellers sell on both platforms in this case.

If P1 commits not to enter, (54), Lemmas 13 and 15 imply that both sellers sell on P1 regardless of P2's commitment.

Consequently, in equilibrium, P1 makes no commitment and P2 commits not to enter, and both sellers sell on both platforms if $\sigma \in \left(1+\frac{\Theta_2}{\Theta_1}, 1+\frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2[8+\Omega_j]^2[\widetilde{\Delta}_{22}]^2}\right)$.

<u>Case IV</u>**.** $\sigma \in \left(1+\frac{\Theta_2[\Delta_{21}]^2}{\Theta_1[\Delta_{11}]^2}, 1+\frac{\Theta_2}{\Theta_1}\right)$. Lemma 18 implies that:

$$1+\frac{16\Theta_1\,[\,1-\Omega_1\,]\,[\,(\,2+\Omega_1\,)\;\Delta_{11}\,]^2}{\Theta_2\,[\,8+\Omega_1\,]^2\left[\,\widetilde{\Delta}_{21}\,\right]^2} > 1+\frac{16\Theta_1\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{12}\,]^2}{\Theta_2\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{22}\,\right]^2} > 1+\frac{\Theta_2}{\Theta_1} > \sigma > 1+\frac{\Theta_2\,[\Delta_{21}]^2}{\Theta_1\,[\Delta_{11}]^2}$$

$$> 1+\frac{\Theta_2\,[\Delta_{22}]^2}{\Theta_1\,[\Delta_{12}]^2} > 1+\frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{21}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{11}\,\right]^2} > 1+\frac{16\Theta_2\,[\,1-\Omega_j\,]\,[\,(\,2+\Omega_j\,)\;\Delta_{22}\,]^2}{\Theta_1\,[\,8+\Omega_j\,]^2\left[\,\widetilde{\Delta}_{12}\,\right]^2}. \tag{55}$$

If P2 makes no commitment, (55), Lemmas 14 and 15 imply that both sellers sell on P1. Lemmas 11 and 12 imply that: (i) P1's profit is $\sigma\Theta_1\,M_{11}-F+\sigma\Theta_1\,M_{12}-F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\sigma\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\sigma\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1\,M_{1j}-F > \frac{\Theta_1[\widetilde{\Delta}_{1j}]^2}{8\,b_j^S}$, i.e., P1 secures more profit by making no commitment than by committing to no entry. Therefore, if P2 makes no commitment, then P1 makes no commitment, and both S1 and S2 sell on P1 in this case.

If P2 commits not to enter, (55), Lemmas 13 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P1's profit is: (i) $\Theta_1\,M_{11}-F+\Theta_1\,M_{12}-F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1\,M_{1j}-F > \frac{\Theta_1[\widetilde{\Delta}_{1j}]^2}{8\,b_j^S}$, which implies that $\Theta_1\,M_{11}-F+\Theta_1\,M_{12}-F > \frac{\Theta_1[\widetilde{\Delta}_{11}]^2}{8\,b_1^S} + \frac{\Theta_1[\widetilde{\Delta}_{12}]^2}{8\,b_2^S}$. Therefore, if P2 commits not to enter, P1 makes no commitment, and each seller sells on both platforms in this case.

If P1 makes no commitment, (55), Lemmas 14 and 16 imply that both sellers: (i) sell on P1 if P2 makes no commitment; and (ii) sell on both platforms if P2 commits not to enter. Lemmas 11 and 12 imply that P2's profit is: (i) zero if P2 makes no commitment; and (ii) $\frac{\Theta_2[\widetilde{\Delta}_{21}]^2}{8\,b_1^S} + \frac{\Theta_2[\widetilde{\Delta}_{22}]^2}{8\,b_2^S}$ if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment. Consequently, if P1 makes no commitment, then P2 commits not to enter, and both sellers sell on both platforms in this case.

If P1 commits not to enter, (55), Lemmas 13 and 15 imply that: (i) each seller sells on both platforms if P2 commits not to enter; and (ii) sells on P1 if P2 makes no commitment. Lemmas 11 and 12 imply that P2's profit is: (i) zero if P2 makes no commitment; and (ii) $\frac{\Theta_2 \left[\widetilde{\Delta}_{21} \right]^2}{8\, b_1^S} + \frac{\Theta_2 \left[\widetilde{\Delta}_{22} \right]^2}{8\, b_2^S}$ if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment. Consequently, if P1 commits not to enter, then P2 commits not to enter, and both sellers sell on both platforms in this case.

Consequently, in equilibrium, P1 makes no commitment and P2 commits not to enter, and both sellers sell on both platforms if $\sigma \in \left(1 + \frac{\Theta_2 [\Delta_{21}]^2}{\Theta_1 [\Delta_{11}]^2}, 1 + \frac{\Theta_2}{\Theta_1} \right)$.

<u>Case V</u>**.** $\sigma \in \left(1 + \frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2}, 1 + \frac{\Theta_2 [\Delta_{21}]^2}{\Theta_1 [\Delta_{11}]^2} \right)$. Lemma 18 implies that:

$$1 + \frac{16 \Theta_1 \left[1 - \Omega_1 \right] \left[\left(2 + \Omega_1 \right) \Delta_{11} \right]^2}{\Theta_2 \left[8 + \Omega_1 \right]^2 \left[\widetilde{\Delta}_{21} \right]^2} > 1 + \frac{16 \Theta_1 \left[1 - \Omega_j \right] \left[\left(2 + \Omega_j \right) \Delta_{12} \right]^2}{\Theta_2 \left[8 + \Omega_j \right]^2 \left[\widetilde{\Delta}_{22} \right]^2} > 1 + \frac{\Theta_2}{\Theta_1} > 1 + \frac{\Theta_2 \left[\Delta_{21} \right]^2}{\Theta_1 \left[\Delta_{11} \right]^2} > \sigma$$

$$> 1 + \frac{\Theta_2 \left[\Delta_{22} \right]^2}{\Theta_1 \left[\Delta_{12} \right]^2} > 1 + \frac{16 \Theta_2 \left[1 - \Omega_j \right] \left[\left(2 + \Omega_j \right) \Delta_{21} \right]^2}{\Theta_1 \left[8 + \Omega_j \right]^2 \left[\widetilde{\Delta}_{11} \right]^2} > 1 + \frac{16 \Theta_2 \left[1 - \Omega_j \right] \left[\left(2 + \Omega_j \right) \Delta_{22} \right]^2}{\Theta_1 \left[8 + \Omega_j \right]^2 \left[\widetilde{\Delta}_{12} \right]^2}. \tag{56}$$

If P2 makes no commitment, (56), Lemmas 14 and 15 imply that: (i) S1 sells on both platforms and S2 sells on P1 if P1 makes no commitment; and (ii) both sellers sell on P1 if P1 commits not to enter. Lemmas 11 and 12 imply that: (i) P1's profit is $\Theta_1 M_{11} - F + \sigma \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} + \frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{11} - F > \frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} + \frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S} > \frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S}$, and $\sigma \Theta_1 M_{12} - F > \frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}$, which implies that $\Theta_1 M_{11} - F + \sigma \Theta_1 M_{12} - F > \frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} + \frac{\sigma \Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}$. Therefore, if P2 makes no commitment, then P1 makes no commitment, and S1 sells on both platforms and S2 sells on P1 in this case.

If P2 commits not to enter, (56), Lemmas 13 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P1's profit is: (i) $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} + \frac{\Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1 \left[\widetilde{\Delta}_{1j} \right]^2}{8\, b_j^S}$, which implies that $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F > \frac{\Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} + \frac{\Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}$. Therefore, if P2 commits not to enter, P1 makes no commitment, and each seller sells on both platforms in this case.

If P1 makes no commitment, (56), Lemmas 14 and 16 imply that: (i) S1 sells on both platforms and S2 sells on P1 if P2 makes no commitment; and (ii) each seller sells on both platforms if P2 commits not to enter. Lemmas 11 and 12 imply that P2's profit is: (i) $\Theta_2 M_{21} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Condition FR ensures that $\Theta_2 M_{21} - F > \frac{\sigma\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\sigma\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S} > \frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$, i.e., P2 secures more profit by making no commitment than by committing not to enter. Consequently, if P1 makes no commitment, then P2 makes no commitment, and S1 sells on both platforms and S2 sells on P1 in this case.

If P1 commits not to enter, (56), Lemmas 13 and 15 imply that: (i) each seller sells on both platforms if P2 commits not to enter; and (ii) sells on P1 if P2 makes no commitment. Lemmas 11 and 12 imply that P2's profit is: (i) zero if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment. Consequently, if P1 commits not to enter, then P2 commits not to enter, and both sellers sell on both platforms in this case.

Consequently, in equilibrium, both platforms make no commitments, and S1 sells on both platforms and S2 sells on P1 if $\sigma \in \left(1 + \frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2}, 1 + \frac{\Theta_2 [\Delta_{21}]^2}{\Theta_1 [\Delta_{11}]^2}\right)$.

<u>Case VI</u>**.** $\sigma \in \left(1 + \frac{16\Theta_2 [1-\Omega_j][(2+\Omega_j)\Delta_{21}]^2}{\Theta_1 [8+\Omega_j]^2 [\widetilde{\Delta}_{11}]^2}, 1 + \frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2}\right)$. Lemma 18 implies that:

$$1+\frac{16\Theta_1 [1-\Omega_1][(2+\Omega_1)\Delta_{11}]^2}{\Theta_2 [8+\Omega_1]^2 \left[\widetilde{\Delta}_{21}\right]^2} > 1+\frac{16\Theta_1 [1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2 [8+\Omega_j]^2 \left[\widetilde{\Delta}_{22}\right]^2} > 1+\frac{\Theta_2}{\Theta_1} > 1+\frac{\Theta_2 [\Delta_{21}]^2}{\Theta_1 [\Delta_{11}]^2}$$

$$> 1+\frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2} > \sigma > 1+\frac{16\Theta_2 [1-\Omega_j][(2+\Omega_j)\Delta_{21}]^2}{\Theta_1 [8+\Omega_j]^2 \left[\widetilde{\Delta}_{11}\right]^2} > 1+\frac{16\Theta_2 [1-\Omega_j][(2+\Omega_j)\Delta_{22}]^2}{\Theta_1 [8+\Omega_j]^2 \left[\widetilde{\Delta}_{12}\right]^2}. \tag{57}$$

If P2 makes no commitment, (57), Lemmas 14 and 15 imply that: (i) each seller sells on both platforms if P1 makes no commitment; and (ii) both sellers sell on P1 if P1 commits not to enter. Lemmas 11 and 12 imply that: (i) P1's profit is $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\sigma\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{11} - F > \frac{\sigma\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$, which implies that $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F > \frac{\sigma\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$. Therefore, if P2 makes no commitment, then P1 makes no commitment, and each seller sells on both platforms in this case.

If P2 commits not to enter, (57), Lemmas 13 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P1's profit is: (i) $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter. Condition

FR ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1 [\widetilde{\Delta}_{1j}]^2}{8 b_j^S}$, which implies that $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F > \frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$. Therefore, if P2 commits not to enter, P1 makes no commitment, and each seller sells on both platforms in this case.

If P1 makes no commitment, (57), Lemmas 14 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P2's profit is: (i) $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Condition FR ensures that $\Theta_2 M_{21} - F > \frac{\sigma \Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\sigma \Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S} > \frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$, which implies that $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F > \frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$. Consequently, if P1 makes no commitment, then P2 makes no commitment, and each seller sells on both platforms in this case.

If P1 commits not to enter, (57), Lemmas 13 and 15 imply that each seller: (i)sells on both platforms if P2 commits not to enter; and (ii) sells on P1 if P2 makes no commitment. Lemmas 11 and 12 imply that P2's profit is: (i) zero if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment. Consequently, if P1 commits not to enter, then P2 commits not to enter, and both sellers sell on both platforms in this case.

Consequently, in equilibrium, both platforms make no commitments, and each seller sells on both platforms if $\sigma \in \left(1 + \frac{16 \Theta_2 [1 - \Omega_j] [(2 + \Omega_j) \Delta_{21}]^2}{\Theta_1 [8 + \Omega_j]^2 [\widetilde{\Delta}_{11}]^2}, 1 + \frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2}\right)$.

<u>Case VII</u>**.** $\sigma \in \left(1 + \frac{16 \Theta_2 [1 - \Omega_j] [(2 + \Omega_j) \Delta_{22}]^2}{\Theta_1 [8 + \Omega_j]^2 [\widetilde{\Delta}_{12}]^2}, 1 + \frac{16 \Theta_2 [1 - \Omega_j] [(2 + \Omega_j) \Delta_{21}]^2}{\Theta_1 [8 + \Omega_j]^2 [\widetilde{\Delta}_{11}]^2}\right)$. Lemma 18 implies that:

$$1 + \frac{16 \Theta_1 [1 - \Omega_1] [(2 + \Omega_1) \Delta_{11}]^2}{\Theta_2 [8 + \Omega_1]^2 \left[\widetilde{\Delta}_{21}\right]^2} > 1 + \frac{16 \Theta_1 [1 - \Omega_j] [(2 + \Omega_j) \Delta_{12}]^2}{\Theta_2 [8 + \Omega_j]^2 \left[\widetilde{\Delta}_{22}\right]^2} > 1 + \frac{\Theta_2}{\Theta_1} > 1 + \frac{\Theta_2 [\Delta_{21}]^2}{\Theta_1 [\Delta_{11}]^2}$$

$$> 1 + \frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2} > 1 + \frac{16 \Theta_2 [1 - \Omega_j] [(2 + \Omega_j) \Delta_{21}]^2}{\Theta_1 [8 + \Omega_j]^2 \left[\widetilde{\Delta}_{11}\right]^2} > \sigma > 1 + \frac{16 \Theta_2 [1 - \Omega_j] [(2 + \Omega_j) \Delta_{22}]^2}{\Theta_1 [8 + \Omega_j]^2 \left[\widetilde{\Delta}_{12}\right]^2}. \tag{58}$$

If P2 makes no commitment, (58), Lemmas 14 and 15 imply that: (i) each seller sells on both platforms if P1 makes no commitment; and (ii) S1 sells on both platforms and S2 sells on P1 if P1 commits not to enter. Lemmas 11 and 12 imply that: (i) P1's profit is $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma \Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{11} - F > \frac{\sigma \Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma \Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S} > \frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma \Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$, which implies that $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F > \frac{\sigma \Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma \Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$.

Therefore, if P2 makes no commitment, then P1 makes no commitment, and each seller sells on both platforms in this case.

If P2 commits not to enter, (58), Lemmas 13 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P1's profit is: (i) $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1 [\widetilde{\Delta}_{1j}]^2}{8 b_j^S}$, which implies that $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F > \frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$. Therefore, if P2 commits not to enter, P1 makes no commitment, and each seller sells on both platforms in this case.

If P1 makes no commitment, (58), Lemmas 14 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P2's profit is: (i) $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Condition FR ensures that $\Theta_2 M_{21} - F > \frac{\sigma\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\sigma\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S} > \frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$, which implies that $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F > \frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$. Consequently, if P1 makes no commitment, then P2 makes no commitment, and each seller sells on both platforms in this case.

If P1 commits not to enter, (58), Lemmas 13 and 15 imply that: (i) each seller sells on both platforms if P2 commits not to enter; and (ii) S1 sells on both platforms and S2 sells on P1 if P2 makes no commitment. Lemmas 11 and 12 imply that P2's profit is: (i) $\Theta_2 M_{21} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Condition FR ensures that $\Theta_2 M_{21} - F > \frac{\sigma\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\sigma\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S} > \frac{\Theta_2 [\widetilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\widetilde{\Delta}_{22}]^2}{8 b_2^S}$,. Consequently, if P1 commits not to enter, then P2 commits not to enter, and S1 sells on both platforms and S2 sells on P1 in this case.

Consequently, in equilibrium, both platforms make no commitments, each seller sells on both platforms if $\sigma \in \left(1 + \frac{16\Theta_2[1-\Omega_j][(2+\Omega_j)\Delta_{22}]^2}{\Theta_1[8+\Omega_j]^2[\widetilde{\Delta}_{12}]^2}, 1 + \frac{16\Theta_2[1-\Omega_j][(2+\Omega_j)\Delta_{21}]^2}{\Theta_1[8+\Omega_j]^2[\widetilde{\Delta}_{11}]^2}\right)$.

<u>Case VIII</u>. $\sigma \in \left(1, 1 + \frac{16\Theta_2[1-\Omega_j][(2+\Omega_j)\Delta_{22}]^2}{\Theta_1[8+\Omega_j]^2[\widetilde{\Delta}_{12}]^2}\right)$. Lemma 18 implies that:

$$1+\frac{16\Theta_1 [1-\Omega_1][(2+\Omega_1)\Delta_{11}]^2}{\Theta_2 [8+\Omega_1]^2 \left[\widetilde{\Delta}_{21}\right]^2} > 1+\frac{16\Theta_1 [1-\Omega_j][(2+\Omega_j)\Delta_{12}]^2}{\Theta_2 [8+\Omega_j]^2 \left[\widetilde{\Delta}_{22}\right]^2} > 1+\frac{\Theta_2}{\Theta_1} > 1+\frac{\Theta_2 [\Delta_{21}]^2}{\Theta_1 [\Delta_{11}]^2}$$

$$> 1+\frac{\Theta_2 [\Delta_{22}]^2}{\Theta_1 [\Delta_{12}]^2} > 1+\frac{16\Theta_2 [1-\Omega_j][(2+\Omega_j)\Delta_{21}]^2}{\Theta_1 [8+\Omega_j]^2 \left[\widetilde{\Delta}_{11}\right]^2} > 1+\frac{16\Theta_2 [1-\Omega_j][(2+\Omega_j)\Delta_{22}]^2}{\Theta_1 [8+\Omega_j]^2 \left[\widetilde{\Delta}_{12}\right]^2} > \sigma. \tag{59}$$

If P2 makes no commitment, (59), Lemmas 14 and 15 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that: (i) P1's profit is $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\Theta_1 [\tilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\tilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{11} - F > \frac{\sigma\Theta_1 [\tilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma\Theta_1 [\tilde{\Delta}_{12}]^2}{8 b_2^S} > \frac{\Theta_1 [\tilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma\Theta_1 [\tilde{\Delta}_{12}]^2}{8 b_2^S}$, which implies that $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F > \frac{\sigma\Theta_1 [\tilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\sigma\Theta_1 [\tilde{\Delta}_{12}]^2}{8 b_2^S}$. Therefore, if P2 makes no commitment, then P1 makes no commitment, and each seller sells on both platforms in this case.

If P2 commits not to enter, (59), Lemmas 13 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P1's profit is: (i) $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1 [\tilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\tilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter. Condition FR ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1 [\tilde{\Delta}_{1j}]^2}{8 b_j^S}$, which implies that $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F > \frac{\Theta_1 [\tilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\tilde{\Delta}_{12}]^2}{8 b_2^S}$. Therefore, if P2 commits not to enter, P1 makes no commitment, and each seller sells on both platforms in this case.

If P1 makes no commitment, (59), Lemmas 14 and 16 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P2's profit is: (i) $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Condition FR ensures that $\Theta_2 M_{21} - F > \frac{\sigma\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\sigma\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S} > \frac{\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S}$, which implies that $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F > \frac{\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S}$. Consequently, if P1 makes no commitment, then P2 makes no commitment, and each seller sells on both platforms in this case.

If P1 commits not to enter, (59), Lemmas 13 and 15 imply that each seller sells on both platforms. Lemmas 11 and 12 imply that P2's profit is: (i) $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S}$ if P2 commits not to enter. Condition FR ensures that $\Theta_2 M_{21} - F > \frac{\sigma\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\sigma\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S} > \frac{\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S}$, which implies that $\Theta_2 M_{21} - F + \Theta_2 M_{22} - F > \frac{\Theta_2 [\tilde{\Delta}_{21}]^2}{8 b_1^S} + \frac{\Theta_2 [\tilde{\Delta}_{22}]^2}{8 b_2^S}$. Consequently, if P1 commits not to enter, then P2 makes no commitment, and each seller sells on both platforms in this case.

Consequently, in equilibrium, both platforms make no commitments, each seller sells on both platforms if $\sigma \in \left(1, 1 + \frac{16\Theta_2 [1-\Omega_j][(2+\Omega_j)\Delta_{22}]^2}{\Theta_1 [8+\Omega_j]^2 [\tilde{\Delta}_{12}]^2}\right)$. ■

Proposition 4. *Suppose* $\frac{\Theta_1}{\Theta_2} \in \left(\sqrt{\phi_{1j}}, \phi_{1j}\right)$ *and Assumption 1 holds. Further suppose* $\Omega_1 = \Omega_2 \in (0.5, 1)$ *and platform entry is feasible (Condition FR holds). Then* $CS < CS^M$ *if* $\sigma \in (1 + \frac{16\Theta_1 [1-\Omega_j][(2+\Omega_j)\Delta_{11}]^2}{\Theta_2 [8+\Omega_j]^2 [\tilde{\Delta}_{21}]^2}, \frac{\varsigma_{11} + \varsigma_{12}}{\frac{1}{2b_1^S}\left[\frac{\tilde{\Delta}_{11}}{4}\right]^2 + \frac{1}{2b_2^S}\left[\frac{\tilde{\Delta}_{12}}{4}\right]^2})$.

Proof. In the presence of platform entry, consumer surplus when consuming Pk's product and Sj's product is given by

$$CS = U\left(Q_{kj}^P, Q_{kj}^S\right) - p_{kj}^P Q_{kj}^P - p_{kj}^S Q_{kj}^S$$

$$= d_{1kj}\, Q_{kj}^P + d_{2kj}\, Q_{kj}^S - \frac{1}{2}\left[d_{3kj}\left(Q_{kj}^P\right)^2 + 2\, d_{4kj}\, Q_{kj}^P\, Q_{kj}^S + d_{5kj}\left(Q_{kj}^S\right)^2\right] - p_{kj}^P Q_{kj}^P - p_{kj}^S Q_{kj}^S. \quad (60)$$

(60) implies inverse demands are given by:

$$p_{kj}^P = d_{1kj} - d_{3kj}\, Q_{kj}^P - d_{4kj}\, Q_{kj}^S, \quad (61)$$

$$p_{kj}^S = d_{2kj} - d_{4kj}\, Q_{kj}^P - d_{5kj}\, Q_{kj}^S. \quad (62)$$

(1) and (2) imply that:

$$p_{kj}^P = \frac{\eta_j \alpha_j + \beta_j^S \theta_j \alpha_j}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2} - \frac{\beta_j^S}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2} q_{kj}^P - \frac{\eta_j}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2} q_{kj}^S \quad (63)$$

$$p_{kj}^S = \frac{\eta_j \theta_j \alpha_j + \beta_j^P \alpha_j}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2} - \frac{\eta_j}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2} q_{kj}^P - \frac{\beta_j^P}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2} q_{kj}^S \quad (64)$$

(61) - (64) imply that:

$$d_{1kj} = \frac{\eta_j \alpha_j + \beta_j^S \theta_j \alpha_j}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2},\ d_{2kj} = \frac{\eta_j \theta_j \alpha_j + \beta_j^P \alpha_j}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2},\ d_{3kj} = \frac{1}{\sigma \Theta_k} \frac{\beta_j^S}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2},$$

$$d_{4kj} = \frac{1}{\sigma \Theta_k} \frac{\eta_j}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2},\ d_{5kj} = \frac{1}{\sigma \Theta_k} \frac{\beta_j^P}{\beta_j^S\, \beta_j^P - \left[\eta_j\right]^2}, \quad (65)$$

where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.

(60) implies that:

$$\begin{aligned}
CS \;=\;& d_{1kj}\, Q_{kj}^P + d_{2kj}\, Q_{kj}^S - \frac{1}{2}\, d_{3kj}\left(Q_{kj}^P\right)^2 - d_{4kj}\, Q_{kj}^P\, Q_{kj}^S - \frac{1}{2} d_{5kj}\left(Q_{kj}^S\right)^2 \\
& - p_{kj}^P Q_{kj}^P - p_{kj}^S Q_{kj}^S. \\
=\;& d_{1kj}\, Q_{kj}^P - d_{3kj}\left(Q_{kj}^P\right)^2 + \frac{1}{2}\, d_{3kj}\left(Q_{kj}^P\right)^2 - d_{4kj}\, Q_{kj}^P\, Q_{kj}^S \\
& + d_{2kj}\, Q_{kj}^S - d_{5kj}\left(Q_{kj}^S\right)^2 + \frac{1}{2} d_{5kj}\left(Q_{kj}^S\right)^2 - d_{4kj}\, Q_{kj}^P\, Q_{kj}^S \\
& + d_{4kj}\, Q_{kj}^P\, Q_{kj}^S - p_{kj}^P Q_{kj}^P - p_{kj}^S Q_{kj}^S \\
=\;& Q_{kj}^P\left[d_{1kj} - d_{3kj} Q_{kj}^P - d_{4kj}\, Q_{kj}^S\right] + Q_{kj}^S\left[d_{2kj} - d_{5kj} Q_{kj}^S - d_{4kj}\, Q_{kj}^P\right] \\
& + \frac{1}{2}\, d_{3kj}\left(Q_{kj}^P\right)^2 + \frac{1}{2} d_{5kj}\left(Q_{kj}^S\right)^2 + d_{4kj}\, Q_{kj}^P\, Q_{kj}^S - p_{kj}^P Q_{kj}^P - p_{kj}^S Q_{kj}^S. \quad (66)
\end{aligned}$$

(61), (62), and (66) imply that:

$$CS = Q_{kj}^P p_{kj}^P + Q_{kj}^S p_{kj}^S + \frac{1}{2} d_{3kj} \left(Q_{kj}^P\right)^2 + \frac{1}{2} d_{5kj} \left(Q_{kj}^S\right)^2 + d_{4kj} Q_{kj}^P Q_{kj}^S - p_{kj}^P Q_{kj}^P - p_{kj}^S Q_{kj}^S$$

$$= \frac{1}{2} d_{3kj} \left(Q_{kj}^P\right)^2 + \frac{1}{2} d_{5kj} \left(Q_{kj}^S\right)^2 + d_{4kj} Q_{kj}^P Q_{kj}^S. \tag{67}$$

(65) and (67) imply that:

$$CS = \frac{1}{2\sigma\Theta_k} \frac{\beta_j^S}{\beta_j^S \beta_j^P - \left[\eta_j\right]^2} \left[Q_{kj}^P\right]^2 + \frac{1}{2\sigma\Theta_k} \frac{\beta_j^P}{\beta_j^S \beta_j^P - \left[\eta_j\right]^2} \left[Q_{kj}^S\right]^2 + \frac{1}{\sigma\Theta_k} \frac{\eta_j}{\beta_j^S \beta_j^P - \left[\eta_j\right]^2} Q_{kj}^P Q_{kj}^S$$

$$= \frac{\frac{\beta_j^S}{2} \left[Q_{kj}^P\right]^2 + \frac{\beta_j^P}{2} \left[Q_{kj}^S\right]^2 + \eta_j Q_{kj}^P Q_{kj}^S}{\sigma\Theta_k \left[\beta_j^S \beta_j^P - \left(\eta_j\right)^2\right]}, \tag{68}$$

where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.

Lemma 6 and (68) imply that in the presence of platform entry, consumer surplus when consuming Pk's product and Sj's product is:

$$CS = \frac{\frac{\beta_j^S}{2} \left[\frac{\sigma\Theta_k \overline{\Delta}_{kj}}{2} + \frac{\eta_j \sigma\Theta_k (2+\Omega_j) \Delta_{kj}}{2\beta_j^S (8+\Omega_j)}\right]^2 + \frac{\beta_j^P}{2} \left[\frac{\sigma\Theta_k [2+\Omega_j] \Delta_{kj}}{8+\Omega_j}\right]^2 + \eta_j \left[\frac{\sigma\Theta_k \overline{\Delta}_{kj}}{2} + \frac{\eta_j \sigma\Theta_k [2+\Omega_j] \Delta_{kj}}{2\beta_j^S [8+\Omega_j]}\right] \frac{\sigma\Theta_k [2+\Omega_j] \Delta_{kj}}{8+\Omega_j}}{\sigma\Theta_k \left[\beta_j^S \beta_j^P - \left(\eta_j\right)^2\right]}$$

$$= \sigma\Theta_k \frac{\frac{\beta_j^S}{2} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j (2+\Omega_j) \Delta_{kj}}{2\beta_j^S (8+\Omega_j)}\right]^2 + \frac{\beta_j^P}{2} \left[\frac{[2+\Omega_j] \Delta_{kj}}{8+\Omega_j}\right]^2 + \eta_j \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j [2+\Omega_j] \Delta_{kj}}{2\beta_j^S [8+\Omega_j]}\right] \frac{[2+\Omega_j] \Delta_{kj}}{8+\Omega_j}}{\beta_j^S \beta_j^P - \left[\eta_j\right]^2} = \sigma\Theta_k \varsigma_{kj}, \tag{69}$$

where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk. The last equality in (69) reflects (8).

(60) implies that if Sj sells on Pk and Pk does not enter Sj's product market, then $Q_{kj}^P = 0$. Thus, consumer surplus when consuming Sj's product in the absence of platform entry is given by

$$CS = d_{2kj} Q_{kj}^S - \frac{d_{5kj}}{2} \left[Q_{kj}^S\right]^2 - p_{kj}^S Q_{kj}^S. \tag{70}$$

(70) implies that the inverse demand is given by:

$$p_{kj}^S = d_{2kj} - d_{5kj} Q_{kj}^S. \tag{71}$$

(3) implies that:

$$p_{kj}^S = \frac{A_j}{b_j^S} - \frac{1}{b_j^S} Q_{kj}^S. \tag{72}$$

(71) and (72) imply that:

$$d_{2kj} = \frac{A_j}{b_j^S}, \text{ and } d_{5kj} = \frac{1}{\sigma \Theta_k b_j^S}, \tag{73}$$

where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.

(70) implies that:

$$CS \;=\; Q_{kj}^S \left[d_{2kj} - \frac{d_{5kj}}{2} Q_{kj}^S \right] - p_{kj}^S Q_{kj}^S \;=\; Q_{kj}^S \left[d_{2kj} - d_{5kj}\, Q_{kj}^S \right] + \frac{d_{5kj}}{2} \left[Q_{kj}^S \right]^2 - p_{kj}^S Q_{kj}^S$$

$$= \; p_{kj}^S Q_{kj}^S + \frac{d_{5kj}}{2} \left[Q_{kj}^S \right]^2 - p_{kj}^S Q_{kj}^S = \frac{d_{5kj}}{2} \left[Q_{kj}^S \right]^2 = \frac{1}{2 \sigma \Theta_k b_j^S} \left[Q_{kj}^S \right]^2, \tag{74}$$

where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.The first equality in (74) reflects (71), and the last equality in (74) reflects (73).

Lemma 3 and (74) imply that consumer surplus when consuming Sj's product in the absence of platform entry is:

$$CS = \frac{1}{2 \sigma \Theta_k b_j^S} \left[\frac{\sigma \Theta_k \, \widetilde{\Delta}_{kj}}{4} \right]^2 = \frac{\sigma \Theta_k}{2 b_j^S} \left[\frac{\widetilde{\Delta}_{kj}}{4} \right]^2, \tag{75}$$

where $\sigma = 1$ if Sj sells on both platforms and $\sigma > 1$ if Sj sells exclusively on Pk.

Proposition 1 implies that when P1 is the monopoly platform, then in equilibrium, each seller competes against P1 under MP. P1 does not provide the exogenous boost σ under MP. Therefore, (69) implies that consumer surplus under MP is:

$$CS^M \;=\; \Theta_1 \, \varsigma_{11} + \Theta_1 \, \varsigma_{12}, \tag{76}$$

Since $\sigma \in \left(1 + \frac{16 \Theta_1 [1 - \Omega_j] [(2 + \Omega_j) \Delta_{11}]^2}{\Theta_2 [8 + \Omega_j]^2 \left[\widetilde{\Delta}_{21} \right]^2}, \frac{\varsigma_{11} + \varsigma_{12}}{\frac{1}{2 b_1^S} \left[\frac{\widetilde{\Delta}_{11}}{4} \right]^2 + \frac{1}{2 b_2^S} \left[\frac{\widetilde{\Delta}_{12}}{4} \right]^2} \right)$, Proposition 3 shows that both P1 and P2 commit not to enter, and both sellers sell on P1. (75) implies that consumer surplus is:

$$CS = \frac{\sigma \Theta_1}{2 b_1^S} \left[\frac{\widetilde{\Delta}_{11}}{4} \right]^2 + \frac{\sigma \Theta_1}{2 b_2^S} \left[\frac{\widetilde{\Delta}_{12}}{4} \right]^2,$$

where $\sigma > 1$.

Observe that:

$$CS < CS^M \Leftrightarrow \frac{\sigma \Theta_1}{2 b_1^S} \left[\frac{\widetilde{\Delta}_{11}}{4} \right]^2 + \frac{\sigma \Theta_1}{2 b_2^S} \left[\frac{\widetilde{\Delta}_{12}}{4} \right]^2 < \Theta_1 \, \varsigma_{11} + \Theta_1 \, \varsigma_{12}$$

$$\Leftrightarrow \sigma < \frac{\varsigma_{11} + \varsigma_{12}}{\frac{1}{2 b_1^S} \left[\frac{\widetilde{\Delta}_{11}}{4} \right]^2 + \frac{1}{2 b_2^S} \left[\frac{\widetilde{\Delta}_{12}}{4} \right]^2}. \quad \blacksquare$$

Proposition 4 shows that when the incumbent is a stronger platform (i.e., $\frac{\Theta_1}{\Theta_2} \in \left(\sqrt{\phi_{1j}}, \phi_{1j}\right)$ where $\sqrt{\phi_{1j}} > 1$) but a weaker seller (i.e., $c^P_{1j} > c^P_{2j}$) than the entrant, increased platform competition can reduce consumer surplus if the single-homing rewards offered by platforms are high enough to promote single-homing (i.e., $\sigma > 1 + \frac{16\Theta_1[1-\Omega_j][(2+\Omega_j)\Delta_{11}]^2}{\Theta_2[8+\Omega_j]^2\left[\widetilde{\Delta}_{21}\right]^2}$), yet insufficient to offset the negative effects of decreased competition on consumers (i.e., $\sigma < \frac{\varsigma_{11}+\varsigma_{12}}{\frac{1}{2b^S_1}\left[\frac{\widetilde{\Delta}_{11}}{4}\right]^2+\frac{1}{2b^S_2}\left[\frac{\widetilde{\Delta}_{12}}{4}\right]^2}$).

2 Seller entry/competition

In this section, I explore two market categories, labeled as $g \in \{1, 2\}$, where each category features price competition among more than two third-party sellers (i.e., $n > 2$).

If a representative consumer purchases $\mathbf{q} \equiv (q_1, ..., q_n)$ at prices $\mathbf{p} \equiv (p_1, ..., p_n)$, his utility is:

$$U(\mathbf{q}, \mathbf{p}) \;=\; \sum_{i=1}^{n} d_i\, q_i - \frac{1}{2}\left[\sum_{i=1}^{n} (q_i)^2 + 2\,\gamma \sum_{j \neq i} q_i\, q_j\right] - \sum_{i=1}^{n} p_i\, q_i\,. \tag{77}$$

Utility maximization entails:

$$\frac{U(\mathbf{q}, \mathbf{p})}{\partial q_i} \;=\; d_i - q_i - \gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - p_i \;=\; 0\,. \tag{78}$$

(78) implies the inverse demand curve for firm i's product is:

$$P_i(q_i, \mathbf{q}_{-i}) \;=\; d_i - q_i - \gamma \sum_{j \neq i} q_j\,. \tag{79}$$

Let c_i denote firm i's constant unit cost of production (including the input price). Then (79) implies that firm i's profit is:

$$\pi_i(q_i, \mathbf{q}_{-i}) \;=\; [\,P_i(q_i, \mathbf{q}_{-i}) - c_i\,]\, q_i\,. \tag{80}$$

Summing (78) for all firms provides:

$$\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} q_i - \gamma \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \sum_{i=1}^{n} p_i \;=\; 0$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} q_i - \gamma\,[\,n-1\,] \sum_{i=1}^{n} q_i - \sum_{i=1}^{n} p_i \;=\; 0$$

$$\Leftrightarrow \quad d_i - q_i - p_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} d_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \gamma\,[\,n-1\,]\, q_i - \gamma\,[\,n-1\,] \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} p_j \;=\; 0\,. \tag{81}$$

(78) and (81) imply:

$$\gamma \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j + \sum_{\substack{j=1 \\ j \neq i}}^{n} d_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \gamma\,[\,n-1\,]\, q_i - \gamma\,[\,n-1\,] \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} p_j \;=\; 0$$

$$\Leftrightarrow \quad \gamma\,[\,n-1\,]\,q_i \;=\; \sum_{\substack{j=1 \\ j\neq i}}^{n} (\,d_j - p_j\,) - [\,1+\gamma\,(\,n-2\,)\,] \sum_{\substack{j=1 \\ j\neq i}}^{n} q_j\,. \tag{82}$$

(78) and (82) imply that for $i = 1, ..., n$:

$$\gamma\,[\,n-1\,]\,q_i \;=\; \sum_{\substack{j=1 \\ j\neq i}}^{n} (\,d_j - p_j\,) - [\,1+\gamma\,(\,n-2\,)\,]\,\frac{1}{\gamma}\,[\,d_i - q_i - p_i\,]$$

$$\Leftrightarrow \quad \left[\gamma\,(\,n-1\,) - \frac{1+\gamma\,(\,n-2\,)}{\gamma}\right] q_i \;=\; \sum_{\substack{j=1 \\ j\neq i}}^{n} (\,d_j - p_j\,) - \frac{1+\gamma\,[\,n-2\,]}{\gamma}\;[\,d_i - p_i\,]$$

$$\Leftrightarrow \quad \left[\frac{\gamma^2\,(\,n-1\,) - 1 - \gamma\,(\,n-2\,)}{\gamma}\right] q_i \;=\; \frac{1}{\gamma}\left[\gamma \sum_{\substack{j=1 \\ j\neq i}}^{n} (\,d_j - p_j\,) - (\,1+\gamma\,[\,n-2\,]\,)\,(\,d_i - p_i\,)\right]$$

$$\Leftrightarrow \quad q_i^* \;=\; \frac{[\,d_i - p_i\,]\,[\,1+\gamma\,(\,n-2\,)\,] - \gamma \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} (\,d_j - p_j\,)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}\,. \tag{83}$$

In the absence of platform entry.

Suppose third-party seller Sj sells on Platform Pk. If Pk does not enter seller market, then Sj competes with the other $n-1$ third-party sellers in the downstream market. (83) imply that when Sj sells on Pk and Pk does not enter seller market, the initial demand for Sj's product is ($j \in \{1, 2, ..., n\}$):

$$q_{kj}^{S-NE} \;=\; \frac{\left[\,d_j - p_{kj}^{S-NE}\,\right][\,1+\gamma\,(\,n-2\,)\,] - \gamma \sum\limits_{\substack{h=1 \\ h\neq j}}^{n} \left(\,d_h - p_{kh}^{S-NE}\,\right)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}. \tag{84}$$

I define

$$\widetilde{\Delta}_{kj} \;\equiv\; \frac{\left[\,d_j - c_j^S\,\right][\,1+\gamma\,(\,n-2\,)\,] - \gamma \sum\limits_{\substack{h=1 \\ h\neq j}}^{n} \left(\,d_h - c_h^S\,\right)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} \tag{85}$$

to be Sj's "selling strength" on Pk in the absence of platform entry. (84) implies that $\widetilde{\Delta}_{kj}$ represents consumers' demand for Sj's product in the absence of platform entry when each third-party seller prices its product at cost.

Assumption 2. *Third-party sellers in category $g \in \{1, 2\}$ have symmetrical selling strength (i.e., for $l \neq j$, $l, j \in \{1, 2, ..., n\}$), $d_j - c_j^S = d_l - c_l^S = d^{Sg} - c^{Sg}$), and third-party sellers in category 1 are stronger than those in category 2 (i.e., $d^{S1} - c^{S1} > d^{S2} - c^{S2}$).*

Assumption 3. *P1 is a stronger seller than P2 (i.e., $d_1 - c_1^P > d_2 - c_2^P$).*

Assumption 4. *Platforms are stronger sellers than third-party sellers (i.e., $d_2 - c_2^P > d^{S1} - c^{S1}$).*

(85) and Assumption 2 imply that the selling strength of Sj in category g, when selling on Pk, is:

$$\widetilde{\Delta}_{kjg} = \frac{\left[d^{Sg} - c^{Sg} \right] [1 + \gamma (n-2)] - \gamma [n-1] \left(d^{Sg} - c^{Sg} \right)}{[1-\gamma][1+\gamma(n-1)]}$$

$$= \frac{\left[d^{Sg} - c^{Sg} \right] [1 + \gamma (n-2) - \gamma (n-1)]}{[1-\gamma][1+\gamma(n-1)]} = \frac{\left[d^{Sg} - c^{Sg} \right] [1-\gamma]}{[1-\gamma][1+\gamma(n-1)]} = \frac{d^{Sg} - c^{Sg}}{1+\gamma[n-1]}. \tag{86}$$

Assumption 2 and (86) imply that for $l \neq j$, $l, j \in \{1, 2, ..., n\}$), and $g \in \{1, 2\}$,

$$\widetilde{\Delta}_{kjg} = \widetilde{\Delta}_{k\lg} = \widetilde{\Delta}_{kg},\ \widetilde{\Delta}_{k1} > \widetilde{\Delta}_{k2},\ \widetilde{\Delta}_{1g} = \widetilde{\Delta}_{2g}. \tag{87}$$

Assumptions 2, 3, and 4 imply that:

$$d_1 - c_1^P > d_2 - c_2^P > d^{S1} - c^{S1} > d^{S2} - c^{S2}. \tag{88}$$

Lemma 19. *Suppose Assumption 2 holds. Further suppose Sj ($j \in \{1, 2, ..., n\}$), whose product is in category g ($g \in \{1, 2\}$), sells on Pk ($j, k \in \{1, 2\}$) and Pk does not enter Sj's market. Given w_{kg}, Sj's equilibrium output (i.e., sales) (Q_{kj}^{S-NE}) is $\frac{\Theta_k [1+\gamma(n-2)] \left[d^S - c^S - w_{kg}^{NE} \right]}{[1+\gamma(n-1)][2+\gamma(n-3)]}$, and Sj's total profit is $\frac{\Theta_k [1-\gamma][1+\gamma(n-2)] \left[d^S - c^S - w_{kg}^{NE} \right]^2}{[1+\gamma(n-1)][2+\gamma(n-3)]^2}$.*

<u>Proof.</u> Suppose Sj ($j \in \{1, 2, ..., n\}$) whose product lies in category g ($g \in \{1, 2\}$) sells on Pk ($j, k \in \{1, 2\}$) and Pk does not enter Sj's market. (84) implies that Sj's profit is:

$$\pi_{kj}^{S-NE} = \left[p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S \right] \Theta_k \, q_{kj}^{S-NE} \tag{89}$$

$$= \left[p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S \right] \Theta_k \frac{\left[d_j - p_{kj}^{S-NE} \right] [1 + \gamma (n-2)] - \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right)}{[1-\gamma][1+\gamma(n-1)]}. \tag{90}$$

(90) implies that Sj chooses p_{kj}^{S-NE} to maximize π_{kj}^{S-NE}:

$$\frac{\partial \pi_{kj}^{S-NE}}{\partial p_{kj}^{S-NE}} = 0 \Leftrightarrow \frac{\left[d_j - p_{kj}^{S-NE} \right] [1 + \gamma (n-2)] - \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right)}{[1-\gamma][1+\gamma(n-1)]}$$

$$- \frac{[1+\gamma(n-2)] \left[p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S \right]}{[1-\gamma][1+\gamma(n-1)]} = 0$$

$$\Leftrightarrow \quad \left[d_j - p_{kj}^{S-NE} \right] \left[1 + \gamma \left(n - 2 \right) \right] - \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right) = \left[1 + \gamma \left(n - 2 \right) \right] \left[p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S \right]$$

$$\Leftrightarrow \quad \left[1 + \gamma \left(n - 2 \right) \right] \left[d_j + w_{kg}^{NE} + c_j^S \right] - \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right) = 2 \, p_{kj}^{S-NE} \left[1 + \gamma \left(n - 2 \right) \right]$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} = \frac{d_j + w_{kg}^{NE} + c_j^S}{2} - \frac{\gamma \sum\limits_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right)}{2 \left[1 + \gamma \left(n - 2 \right) \right]}. \tag{91}$$

(91) implies:

$$2 \left[1 + \gamma \left(n - 2 \right) \right] p_{kj}^{S-NE} = \left[1 + \gamma \left(n - 2 \right) \right] \left[d_j + w_{kg}^{NE} + c_j^S \right] - \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} d_h + \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} p_{kh}^{S-NE}$$

$$\Leftrightarrow \quad \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} p_{kh}^{S-NE} = 2 \left[1 + \gamma \left(n - 2 \right) \right] p_{kj}^{S-NE} - \left[1 + \gamma \left(n - 2 \right) \right] \left[d_j + w_{kg}^{NE} + c_j^S \right] + \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} d_h$$

$$\Leftrightarrow \quad \sum_{\substack{h=1 \\ h \neq j}}^{n} p_{kh}^{S-NE} = \frac{\left[1 + \gamma \left(n - 2 \right) \right] \left[2 \, p_{kj}^{S-NE} - d_j - w_{kg}^{NE} - c_j^S \right]}{\gamma} + \sum_{\substack{h=1 \\ h \neq j}}^{n} d_h. \tag{92}$$

Summing (91) for all firms provides:

$$\sum_{j=1}^{n} p_{kj}^{S-NE} = \frac{\sum\limits_{j=1}^{n} \left(d_j + w_{kg}^{NE} + c_j^S \right)}{2} - \frac{\gamma \sum\limits_{j=1}^{n} \sum\limits_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right)}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n} p_{kj}^{S-NE} = \frac{n \, w_{kg}^{NE} + \sum\limits_{j=1}^{n} \left(d_j + c_j^S \right)}{2} - \frac{\gamma \left[n - 1 \right] \sum\limits_{j=1}^{n} \left(d_j - p_{kj}^{S-NE} \right)}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n} p_{kj}^{S-NE} - \frac{\gamma \left[n - 1 \right] \sum\limits_{j=1}^{n} p_{kj}^{S-NE}}{2 \left[1 + \gamma \left(n - 2 \right) \right]} = \frac{n \, w_{kg}^{NE} + \sum\limits_{j=1}^{n} \left(d_j + c_j^S \right)}{2} - \frac{\gamma \left[n - 1 \right] \sum\limits_{j=1}^{n} d_j}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n} p_{kj}^{S-NE} \left\{ 1 - \frac{\gamma \left[n - 1 \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]} \right\}$$

$$= \frac{n \, w_{kg}^{NE} + \sum\limits_{j=1}^{n} c_j^S}{2} + \frac{\left[1 + \gamma \left(n - 2 \right) \right] \sum\limits_{j=1}^{n} d_j - \gamma \left[n - 1 \right] \sum\limits_{j=1}^{n} d_j}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n} p_{kj}^{S-NE} \left\{ \frac{2 + 2\,\gamma\,[\,n-2\,] \,-\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \right\}$$

$$= \frac{n\,w_{kg}^{NE} + \sum_{j=1}^{n} c_j^S}{2} + \frac{[\,1+\gamma\,(\,n-2\,)-\gamma\,(\,n-1\,)\,] \sum_{j=1}^{n} d_j}{2\,[\,1+\gamma\,(\,n-2\,)\,]}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n} p_{kj}^{S-NE} \left\{ \frac{2+\gamma\,[\,2\,(\,n-2\,)-(\,n-1\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \right\} = \frac{n\,w_{kg}^{NE} + \sum_{j=1}^{n} c_j^S}{2} + \frac{[\,1-\gamma\,] \sum_{j=1}^{n} d_j}{2\,[\,1+\gamma\,(\,n-2\,)\,]}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n} p_{kj}^{S-NE} \frac{2+\gamma\,[\,n-3\,]}{1+\gamma\,[\,n-2\,]} = n\,w_{kg}^{NE} + \sum_{j=1}^{n} c_j^S + \frac{[\,1-\gamma\,] \sum_{j=1}^{n} d_j}{1+\gamma\,[\,n-2\,]}$$

$$\Leftrightarrow \quad \sum_{j=1}^{n} p_{kj}^{S-NE} = \frac{[\,1+\gamma\,(\,n-2\,)\,] \left[n\,w_{kg}^{NE} + \sum_{j=1}^{n} c_j^S \right]}{2+\gamma\,[\,n-3\,]} + \frac{[\,1-\gamma\,] \sum_{j=1}^{n} d_j}{2+\gamma\,[\,n-3\,]}$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} + \sum_{\substack{h=1 \\ h \neq j}}^{n} p_{kh}^{S-NE} = \frac{[\,1+\gamma\,(\,n-2\,)\,] \left[n\,w_{kg}^{NE} + \sum_{j=1}^{n} c_j^S \right]}{2+\gamma\,[\,n-3\,]} + \frac{[\,1-\gamma\,] \sum_{j=1}^{n} d_j}{2+\gamma\,[\,n-3\,]}. \qquad (93)$$

(92) and (93) imply that:

$$p_{kj}^{S-NE} + \frac{[\,1+\gamma\,(\,n-2\,)\,]\,[\,2\,p_{kj}^{S-NE} - d_j - w_{kg}^{NE} - c_j^S\,]}{\gamma} + \sum_{\substack{h=1 \\ h \neq j}}^{n} d_h$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,] \left[n\,w_{kg}^{NE} + \sum_{j=1}^{n} c_j^S \right]}{2+\gamma\,[\,n-3\,]} + \frac{[\,1-\gamma\,] \sum_{j=1}^{n} d_j}{2+\gamma\,[\,n-3\,]}$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} + \frac{2\,p_{kj}^{S-NE}\,[\,1+\gamma\,(\,n-2\,)\,]}{\gamma}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,] \left[n\,w_{kg}^{NE} + \sum_{j=1}^{n} c_j^S \right]}{2+\gamma\,[\,n-3\,]} + \frac{[\,1-\gamma\,] \sum_{j=1}^{n} d_j}{2+\gamma\,[\,n-3\,]} - \sum_{\substack{h=1 \\ h \neq j}}^{n} d_h + \frac{[\,1+\gamma\,(\,n-2\,)\,]\,[\,d_j + w_{kg}^{NE} + c_j^S\,]}{\gamma}$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} \left\{ 1 + \frac{2\,[\,1+\gamma\,(\,n-2\,)\,]}{\gamma} \right\}$$

$$= \frac{[1+\gamma(n-2)]\, n\, w_{kg}^{NE}}{2+\gamma[n-3]} + \frac{[1+\gamma(n-2)]\, w_{kg}^{NE}}{\gamma} + \frac{[1+\gamma(n-2)] \sum_{j=1}^{n} c_j^S}{2+\gamma[n-3]}$$

$$+ \frac{[1-\gamma] \sum_{j=1}^{n} d_j}{2+\gamma[n-3]} - \sum_{\substack{h=1\\h\neq j}}^{n} d_h + \frac{[1+\gamma(n-2)]\, d_j}{\gamma} + \frac{[1+\gamma(n-2)]\, c_j^S}{\gamma}$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} \left\{ \frac{\gamma + 2[1+\gamma(n-2)]}{\gamma} \right\}$$

$$= w_{kg}^{NE} [1+\gamma(n-2)] \left[\frac{n}{2+\gamma(n-3)} + \frac{1}{\gamma} \right] + \frac{[1+\gamma(n-2)]\, c_j^S}{2+\gamma[n-3]} + \frac{[1+\gamma(n-2)] \sum_{\substack{h=1\\h\neq j}}^{n} c_h^S}{2+\gamma[n-3]}$$

$$+ \frac{[1-\gamma]\, d_j}{2+\gamma[n-3]} + \frac{[1-\gamma] \sum_{\substack{h=1\\h\neq j}}^{n} d_h}{2+\gamma[n-3]} - \sum_{\substack{h=1\\h\neq j}}^{n} d_h + \frac{[1+\gamma(n-2)]\, d_j}{\gamma} + \frac{[1+\gamma(n-2)]\, c_j^S}{\gamma}$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} \frac{2+\gamma[2n-3]}{\gamma}$$

$$= w_{kg}^{NE} [1+\gamma(n-2)] \left[\frac{\gamma n + 2 + \gamma(n-3)}{\gamma[2+\gamma(n-3)]} \right] + c_j^S [1+\gamma(n-2)] \left[\frac{1}{2+\gamma(n-3)} + \frac{1}{\gamma} \right]$$

$$+ \frac{[1+\gamma(n-2)] \sum_{\substack{h=1\\h\neq j}}^{n} c_h^S}{2+\gamma[n-3]} + d_j \left[\frac{1-\gamma}{2+\gamma(n-3)} + \frac{1+\gamma(n-2)}{\gamma} \right] + \sum_{\substack{h=1\\h\neq j}}^{n} d_h \left[\frac{1-\gamma}{2+\gamma(n-3)} - 1 \right]$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} \frac{2+\gamma[2n-3]}{\gamma}$$

$$= w_{kg}^{NE} \frac{[1+\gamma(n-2)][2+\gamma(2n-3)]}{\gamma[2+\gamma(n-3)]} + c_j^S [1+\gamma(n-2)] \frac{\gamma + 2 + \gamma[n-3]}{\gamma[2+\gamma(n-3)]}$$

$$+ \sum_{\substack{h=1\\h\neq j}}^{n} c_h^S \frac{[1+\gamma(n-2)]}{2+\gamma[n-3]} + d_j \left\{ \frac{\gamma(1-\gamma)}{\gamma[2+\gamma(n-3)]} + \frac{[1+\gamma(n-2)][2+\gamma(n-3)]}{\gamma[2+\gamma(n-3)]} \right\}$$

$$+ \sum_{\substack{h=1\\h\neq j}}^{n} d_h \left[\frac{1-\gamma-2-\gamma(n-3)}{2+\gamma(n-3)} \right]$$

$$\Leftrightarrow \quad p_{kj}^{S-NE} \frac{2+\gamma[2n-3]}{\gamma}$$

$$= \; w_{kg}^{NE} \, \frac{[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,2\,n-3\,)\,]}{\gamma\,[\,2+\gamma\,(\,n-3\,)\,]} + c_j^S \, \frac{[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]}{\gamma\,[\,2+\gamma\,(\,n-3\,)\,]} + \sum_{\substack{h=1\\h\neq j}}^{n} c_h^S \frac{[\,1+\gamma\,(\,n-2\,)\,]}{2+\gamma\,[\,n-3\,]}$$

$$+ \; d_j \, \frac{\gamma\,[\,1-\gamma\,] + [\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]}{\gamma\,[\,2+\gamma\,(\,n-3\,)\,]} - \sum_{\substack{h=1\\h\neq j}}^{n} d_h \, \frac{[\,1+\gamma\,(\,n-2\,)\,]}{2+\gamma\,[\,n-3\,]}$$

$$\Leftrightarrow \;\; p_{kj}^{S-NE} \, \frac{2+\gamma\,[\,2\,n-3\,]}{\gamma}$$

$$= \; \frac{1}{\gamma\,[\,2+\gamma\,(\,n-3\,)\,]}$$

$$\cdot \{ \, d_j\,[\,\gamma\,[\,1-\gamma\,] + [\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]\,]$$

$$+ \; [\,1+\gamma\,(\,n-2\,)\,] \left[w_{kg}^{NE}\,[\,2+\gamma\,(\,2\,n-3\,)\,] + c_j^S\,[\,2+\gamma\,(\,n-2\,)\,] + \gamma \sum_{\substack{h=1\\h\neq j}}^{n} \left(c_h^S - d_h \right) \right] \Bigg\}$$

$$\Leftrightarrow \;\; p_{kj}^{S-NE} \; = \; \frac{1}{[\,2+\gamma\,(\,2\,n-3\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]}$$

$$\cdot \{ \, d_j\,[\,\gamma\,[\,1-\gamma\,] + [\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]\,]$$

$$+ \; [\,1+\gamma\,(\,n-2\,)\,] \left[w_{kg}^{NE}\,[\,2+\gamma\,(\,2\,n-3\,)\,] + c_j^S\,[\,2+\gamma\,(\,n-2\,)\,] + \gamma \sum_{\substack{h=1\\h\neq j}}^{n} \left(c_h^S - d_h \right) \right] \Bigg\} . \tag{94}$$

(92) implies that:

$$\sum_{\substack{h=1\\h\neq j}}^{n} p_{kh}^{S-NE} \; = \; \frac{2\,[\,1+\gamma\,(\,n-2\,)\,]}{\gamma}\, p_{kj}^{S-NE} - \frac{[\,1+\gamma\,(\,n-2\,)\,]\,\left[\,d_j + w_{kg}^{NE} + c_j^S\,\right]}{\gamma} + \sum_{\substack{h=1\\h\neq j}}^{n} d_h$$

$$\Leftrightarrow \;\; \sum_{\substack{h=1\\h\neq j}}^{n} d_h - \sum_{\substack{h=1\\h\neq j}}^{n} p_{kh}^{S-NE} \; = \; \frac{[\,1+\gamma\,(\,n-2\,)\,]\,\left[\,d_j + w_{kg}^{NE} + c_j^S\,\right]}{\gamma} - \frac{2\,[\,1+\gamma\,(\,n-2\,)\,]}{\gamma}\, p_{kj}^{S-NE}$$

$$\Leftrightarrow \;\; \gamma \sum_{\substack{h=1\\h\neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right) \; = \; [\,1+\gamma\,(\,n-2\,)\,]\,\left[\,d_j + w_{kg}^{NE} + c_j^S\,\right] - 2\,[\,1+\gamma\,(\,n-2\,)\,]\; p_{kj}^{S-NE}$$

$$\Leftrightarrow \;\; \gamma \sum_{\substack{h=1\\h\neq j}}^{n} \left(d_h - p_{kh}^{S-NE} \right) \; = \; [\,1+\gamma\,(\,n-2\,)\,]\,\left[\,w_{kg}^{NE} + c_j^S + d_j - 2\,p_{kj}^{S-NE}\,\right] . \tag{95}$$

(84) and (95) imply that:

$$q_{kj}^{S-NE} = \frac{\left[d_j - p_{kj}^{S-NE} \right] [1 + \gamma (n - 2)] - [1 + \gamma (n - 2)] \left[w_{kg}^{NE} + c_j^S + d_j - 2 \, p_{kj}^{S-NE} \right]}{[1 - \gamma] [1 + \gamma (n - 1)]}$$

$$= \frac{1 + \gamma [n - 2]}{[1 - \gamma] [1 + \gamma (n - 1)]} \left[d_j - p_{kj}^{S-NE} - \left(w_{kg}^{NE} + c_j^S + d_j - 2 \, p_{kj}^{S-NE} \right) \right]$$

$$= \frac{1 + \gamma [n - 2]}{[1 - \gamma] [1 + \gamma (n - 1)]} \left[p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S \right] . \tag{96}$$

(96) implies that:

$$p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S = \frac{[1 - \gamma] [1 + \gamma (n - 1)]}{1 + \gamma [n - 2]} q_{kj}^{S-NE} . \tag{97}$$

(90) and (97) imply that:

$$\pi_{kj}^{S-NE} = \frac{\Theta_k [1 - \gamma] [1 + \gamma (n - 1)]}{1 + \gamma [n - 2]} \left[q_{kj}^{S-NE} \right]^2 . \tag{98}$$

(94) implies that:

$$p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S$$

$$= \frac{1}{[2 + \gamma (2 n - 3)] [2 + \gamma (n - 3)]}$$

$$\cdot \{ d_j [\gamma [1 - \gamma] + [1 + \gamma (n - 2)] [2 + \gamma (n - 3)]]$$

$$+ \; [1 + \gamma (n - 2)] \left[w_{kg}^{NE} [2 + \gamma (2 n - 3)] + c_j^S [2 + \gamma (n - 2)] + \gamma \sum_{\substack{h = 1 \\ h \neq j}}^{n} \left(c_h^S - d_h \right) \right]$$

$$- \, w_{kg}^{NE} [2 + \gamma (2 n - 3)] [2 + \gamma (n - 3)] - c_j^S [2 + \gamma (2 n - 3)] [2 + \gamma (n - 3)] \}$$

$$= \frac{1}{[2 + \gamma (2 n - 3)] [2 + \gamma (n - 3)]}$$

$$\cdot \left\{ d_j \{ \gamma [1 - \gamma] + [1 + \gamma (n - 2)] [2 + \gamma (n - 3)] \} + [1 + \gamma (n - 2)] \gamma \sum_{\substack{h = 1 \\ h \neq j}}^{n} \left(c_h^S - d_h \right) \right.$$

$$+ \, w_{kg}^{NE} [1 + \gamma (n - 2)] \, [2 + \gamma (2 n - 3)] \, + c_j^S [1 + \gamma (n - 2)] [2 + \gamma (n - 2)]$$

$$- \, w_{kg}^{NE} [2 + \gamma (2 n - 3)] [2 + \gamma (n - 3)] - c_j^S [2 + \gamma (2 n - 3)] [2 + \gamma (n - 3)] \}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Bigg\{ d_j [\gamma[1-\gamma] + [1+\gamma(n-2)][2+\gamma(n-3)]] + [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h\neq j}}^{n} \left(c_h^S - d_h\right)$$

$$+ w_{kg}^{NE}\{1+\gamma(n-2) - [2+\gamma(n-3)]\}[2+\gamma(2n-3)]$$

$$+ c_j^S\{[1+\gamma(n-2)][2+\gamma(n-2)] - [2+\gamma(2n-3)][2+\gamma(n-3)]\}\Bigg\}. \tag{99}$$

Observe that:

$$1+\gamma(n-2) - [2+\gamma(n-3)] = \gamma[n-2-(n-3)] - 1 = \gamma - 1;$$

$$\begin{aligned}
&[1+\gamma(n-2)][2+\gamma(n-2)] - [2+\gamma(2n-3)][2+\gamma(n-3)] \\
&= [1+\gamma(n-2)][2+\gamma(n-3+1)] - [1+\gamma(n-2)+1+\gamma(n-1)][2+\gamma(n-3)] \\
&= [1+\gamma(n-2)][2+\gamma(n-3)+\gamma] - [1+\gamma(n-2)][2+\gamma(n-3)] \\
&\qquad - [1+\gamma(n-1)][2+\gamma(n-3)] \\
&= [1+\gamma(n-2)][2+\gamma(n-3)] + \gamma[1+\gamma(n-2)] \\
&\qquad - [1+\gamma(n-2)][2+\gamma(n-3)] - [1+\gamma(n-1)][2+\gamma(n-3)] \\
&= \gamma[1+\gamma(n-2)] - [1+\gamma(n-1)][2+\gamma(n-3)] \\
&= \gamma[1+\gamma(n-2)] - [1+\gamma(n-2+1)][2+\gamma(n-3)] \\
&= \gamma[1+\gamma(n-2)] - [1+\gamma(n-2)+\gamma][2+\gamma(n-3)] \\
&= \gamma[1+\gamma(n-2)] - [1+\gamma(n-2)][2+\gamma(n-3)] - \gamma[2+\gamma(n-3)] \\
&= \gamma\{1+\gamma(n-2) - [2+\gamma(n-3)]\} - [1+\gamma(n-2)][2+\gamma(n-3)] \\
&= \gamma[1+\gamma(n-2) - 2 - \gamma(n-3)] - [1+\gamma(n-2)][2+\gamma(n-3)] \\
&= \gamma\{\gamma[n-2-(n-3)] - 1\} - [1+\gamma(n-2)][2+\gamma(n-3)] \\
&= \gamma[\gamma-1] - [1+\gamma(n-2)][2+\gamma(n-3)] \\
&= -\gamma[1-\gamma] - [1+\gamma(n-2)][2+\gamma(n-3)].
\end{aligned} \tag{100}$$

(99) and (100) imply that:

$$p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S = \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Bigg\{ d_j \{ \gamma [1-\gamma] + [1+\gamma(n-2)][2+\gamma(n-3)] \} + [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(c_h^S - d_h \right)$$

$$+ w_{kg}^{NE} [\gamma - 1] \, [2+\gamma(2n-3)]$$

$$+ c_j^S \{ -\gamma[1-\gamma] - [1+\gamma(n-2)][2+\gamma(n-3)] \} \Big\}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] \{ \gamma[1-\gamma] + [1+\gamma(n-2)][2+\gamma(n-3)] \}$$

$$+ \; [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(c_h^S - d_h \right) - w_{kg}^{NE} [1-\gamma] \, [2+\gamma(2n-3)] \Bigg\}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] \{ \gamma[1-\gamma] + [1+\gamma(n-2)][1+1+\gamma(n-2-1)] \}$$

$$+ \; [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(c_h^S - d_h \right) - w_{kg}^{NE} [1-\gamma] \, [2+\gamma(2n-3)] \Bigg\}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] \{ \gamma[1-\gamma] + [1+\gamma(n-2)][1+\gamma(n-2)+1-\gamma] \}$$

$$+ \; [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(c_h^S - d_h \right) - w_{kg}^{NE} [1-\gamma] \, [2+\gamma(2n-3)] \Bigg\}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] \left\{ \gamma[1-\gamma] + [1-\gamma][1+\gamma(n-2)] + [1+\gamma(n-2)]^2 \right\}$$

$$+ \; [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(c_h^S - d_h \right) - w_{kg}^{NE} [1-\gamma] \, [2+\gamma(2n-3)] \Bigg\}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] [1-\gamma][1+\gamma(n-2)+\gamma] + \left[d_j - c_j^S \right] [1+\gamma(n-2)]^2$$

$$- [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - c_h^S \right) - w_{kg}^{NE} [1-\gamma] [2+\gamma(2n-3)] \Bigg\}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] [1-\gamma][1+\gamma(n-1)] - w_{kg}^{NE} [1-\gamma] [2+\gamma(2n-3)]$$

$$+ \left[d_j - c_j^S \right] [1+\gamma(n-2)]^2 - [1+\gamma(n-2)]\gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - c_h^S \right) \Bigg\}$$

$$= \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] [1-\gamma][1+\gamma(n-1)] - w_{kg}^{NE} [1-\gamma] [2+\gamma(2n-3)]$$

$$+ [1+\gamma(n-2)] \left[\left[d_j - c_j^S \right] [1+\gamma(n-2)] - \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - c_h^S \right) \right] \Bigg\}. \quad (101)$$

(85) implies:

$$\left[d_j - c_j^S \right] [1+\gamma(n-2)] - \gamma \sum_{\substack{h=1 \\ h \neq j}}^{n} \left(d_h - c_h^S \right) = \widetilde{\Delta}_{kj} [1-\gamma][1+\gamma(n-1)]. \quad (102)$$

(101) and (102) imply that:

$$p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S = \frac{1}{[2+\gamma(2n-3)][2+\gamma(n-3)]}$$

$$\cdot \Big\{ \left[d_j - c_j^S \right] [1-\gamma][1+\gamma(n-1)] - w_{kg}^{NE} [1-\gamma] [2+\gamma(2n-3)]$$

$$+ \widetilde{\Delta}_{kj} [1-\gamma][1+\gamma(n-1)][1+\gamma(n-2)] \Big\}$$

$$= \frac{1-\gamma}{[\,2+\gamma\,(\,2\,n-3\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]}$$

$$\cdot \Big\{ \left[\, d_j - c_j^S \,\right] \, [\,1+\gamma\,(\,n-1\,)\,] - w_{kg}^{NE} \, [\,2+\gamma\,(\,2\,n-3\,)\,]$$

$$+ \widetilde{\Delta}_{kj}\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,] \Big\}$$

$$= \frac{[\,1-\gamma\,] \Big\{ \left[\, d_j - c_j^S \,\right] \, [\,1+\gamma\,(\,n-1\,)\,] + \widetilde{\Delta}_{kj}\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,] \Big\}}{[\,2+\gamma\,(\,2\,n-3\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]}$$

$$- \frac{w_{kg}^{NE}\,[\,1-\gamma\,]}{2+\gamma\,[\,n-3\,]}. \tag{103}$$

Assumption 2, (103), and (86) imply that:

$$p_{kj}^{S-NE} - w_{kg}^{NE} - c_j^S = \frac{[\,1-\gamma\,] \Big\{ \left[\, d^S - c^S \,\right] \, [\,1+\gamma\,(\,n-1\,)\,] + \frac{d^S - c^S}{1+\gamma[\,n-1\,]}\,[\,1+\gamma\,(\,n-1\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,] \Big\}}{[\,2+\gamma\,(\,2\,n-3\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]}$$

$$- \frac{w_{kg}^{NE}\,[\,1-\gamma\,]}{2+\gamma\,[\,n-3\,]}$$

$$= \frac{[\,1-\gamma\,] \left\{ \left[\, d^S - c^S \,\right] \, [\,1+\gamma\,(\,n-1\,)\,] + \left[\, d^S - c^S \,\right] [\,1+\gamma\,(\,n-2\,)\,] \right\}}{[\,2+\gamma\,(\,2\,n-3\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]}$$

$$- \frac{w_{kg}^{NE}\,[\,1-\gamma\,]}{2+\gamma\,[\,n-3\,]}$$

$$= \frac{[\,1-\gamma\,] \left[\, d^S - c^S \,\right] [\,2+\gamma\,(\,2\,n-3\,)\,]}{[\,2+\gamma\,(\,2\,n-3\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]} - \frac{w_{kg}^{NE}\,[\,1-\gamma\,]}{2+\gamma\,[\,n-3\,]}$$

$$= \frac{[\,1-\gamma\,] \left[\, d^S - c^S \,\right] - w_{kg}^{NE}\,[\,1-\gamma\,]}{2+\gamma\,[\,n-3\,]} = \frac{[\,1-\gamma\,] \left[\, d^S - c^S - w_{kg}^{NE} \right]}{2+\gamma\,[\,n-3\,]}. \tag{104}$$

(96) and (104) imply that:

$$q_{kj}^{S-NE} = \frac{1+\gamma\,[\,n-2\,]}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]} \frac{[\,1-\gamma\,] \left[\, d^S - c^S - w_{kg}^{NE} \right]}{2+\gamma\,[\,n-3\,]}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,] \left[\, d^S - c^S - w_{kg}^{NE} \right]}{[\,1+\gamma\,(\,n-1\,)\,]\,[\,2+\gamma\,(\,n-3\,)\,]}. \tag{105}$$

(98) and (105) imply that:

$$\pi_{kj}^{S-NE} = \frac{\Theta_k\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}{1+\gamma\,[\,n-2\,]} \frac{[\,1+\gamma\,(\,n-2\,)\,]^2 \left[\, d^S - c^S - w_{kg}^{NE} \right]^2}{[\,1+\gamma\,(\,n-1\,)\,]^2\,[\,2+\gamma\,(\,n-3\,)\,]^2}$$

$$= \frac{\Theta_k [1-\gamma][1+\gamma(n-2)]\left[d^S - c^S - w_{kg}^{NE}\right]^2}{[1+\gamma(n-1)][2+\gamma(n-3)]^2}. \quad \blacksquare \tag{106}$$

Lemma 20. *Suppose Assumption 2 holds. Further suppose $n > 1$ third-party sellers in category g ($g \in \{1,2\}$) sell on Pk ($k \in \{1,2\}$) and Pk does not enter seller market in category g. Then Pk's profit-maximizing commission for each third-party seller in category g is $w_{kg}^{NE} = \frac{d^S - c^S}{2}$.*

Proof. Lemma 19 implies that Pk chooses w_{kg}^{NE} to:

$$\begin{aligned} \text{Maximize} \;\; \Pi_k^{P-NE} &= w_{kg}^{NE}\,\Theta_k \sum_{j=1}^{n} q_{kj}^{S-NE} + \overline{\Pi}_{kl} \\ &= w_{kg}^{NE}\,\Theta_k \sum_{j=1}^{n} \frac{[1+\gamma(n-2)]\left[d^S - c^S - w_{kg}^{NE}\right]}{[1+\gamma(n-1)][2+\gamma(n-3)]} + \overline{\Pi}_{kl} \\ &= w_{kg}^{NE}\,\Theta_k \frac{n[1+\gamma(n-2)]\left[d^S - c^S - w_{kg}^{NE}\right]}{[1+\gamma(n-1)][2+\gamma(n-3)]} + \overline{\Pi}_{kl} \end{aligned} \tag{107}$$

$$\begin{aligned} \Rightarrow \;\; & \frac{\partial \Pi_k}{\partial w_{kg}^{NE}} = 0 \\ \Leftrightarrow \;\; & \frac{n[1+\gamma(n-2)]\left[d^S - c^S - w_{kg}^{NE}\right]}{[1+\gamma(n-1)][2+\gamma(n-3)]} - w_{kg}^{NE} \frac{n[1+\gamma(n-2)]}{[1+\gamma(n-1)][2+\gamma(n-3)]} = 0 \\ \Leftrightarrow \;\; & \frac{n[1+\gamma(n-2)]\left[d^S - c^S\right]}{[1+\gamma(n-1)][2+\gamma(n-3)]} = w_{kg}^{NE} \frac{2n[1+\gamma(n-2)]}{[1+\gamma(n-1)][2+\gamma(n-3)]} \\ \Leftrightarrow \;\; & w_{kg}^{NE} = \frac{d^S - c^S}{2}, \end{aligned} \tag{108}$$

where $\overline{\Pi}_{kl}$ is the profit that Pk secures from sellers in category l ($l \in \{1,2\}$, $l \neq g$), $\overline{\Pi}_{kl} > 0$ if selllers in category l sell on Pk, and $\overline{\Pi}_{kl} = 0$ if selllers in category l do not on Pk. $\blacksquare$

Lemma 21. *Suppose Assumption 2 holds. Further suppose $n > 1$ third-party sellers in category g ($g \in \{1,2\}$) sell on Pk ($k \in \{1,2\}$) and Pk does not enter seller market in category g. Then each seller's (Sj) equilibrium output (Q_{kj}^{S-NE}) is $\frac{\Theta_k \widetilde{\Delta}_{kj}[1+\gamma(n-2)]}{2[2+\gamma(n-3)]}$, Sj's profit is $\frac{\Theta_k[1-\gamma][1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kj}\right]^2}{4[2+\gamma(n-3)]^2}$, and Pk's profit is $\frac{\Theta_k n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kj}\right]^2}{4[2+\gamma(n-3)]}$.*

Proof. Lemmas 19 and 20 imply that consumers' initial demand for product j is:

$$\begin{aligned} q_{kj}^{S-NE} &= \frac{[1+\gamma(n-2)]\left[d^S - c^S - \frac{d^S - c^S}{2}\right]}{[1+\gamma(n-1)][2+\gamma(n-3)]} \\ &= \frac{[1+\gamma(n-2)]\left[d^S - c^S\right]}{2[1+\gamma(n-1)][2+\gamma(n-3)]} = \frac{\widetilde{\Delta}_{kj}[1+\gamma(n-2)]}{2[2+\gamma(n-3)]}. \end{aligned} \tag{109}$$

The last equality in (109) reflects (86).

Lemmas 19 and 20 imply that Sj's profit is:

$$\pi_{kj}^{S-NE} = \frac{\Theta_k [1-\gamma][1+\gamma(n-2)]\left[d^S - c^S - \frac{d^S-c^S}{2}\right]^2}{[1+\gamma(n-1)][2+\gamma(n-3)]^2}$$

$$= \frac{\Theta_k [1-\gamma][1+\gamma(n-2)]\left[d^S - c^S\right]^2}{4[1+\gamma(n-1)][2+\gamma(n-3)]^2}$$

$$= \frac{\Theta_k [1-\gamma][1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kj}\right]^2}{4[2+\gamma(n-3)]^2}. \quad (110)$$

(110) reflects (86).

Lemma 20 and (109) imply that Pk's profit from charging commissions from sellers in category g is:

$$\Pi_k = w_{kg}^{NE}\,\Theta_k \sum_{j=1}^{n} q_{kj}^{S-NE} = \frac{d^S-c^S}{2}\,\Theta_k \sum_{j=1}^{n} \frac{[1+\gamma(n-2)]\left[d^S-c^S\right]}{2[1+\gamma(n-1)][2+\gamma(n-3)]}$$

$$= \frac{d^S-c^S}{2}\,\Theta_k\,\frac{n[1+\gamma(n-2)]\left[d^S-c^S\right]}{2[1+\gamma(n-1)][2+\gamma(n-3)]}$$

$$= \frac{\Theta_k n[1+\gamma(n-2)]\left[d^S-c^S\right]^2}{4[1+\gamma(n-1)][2+\gamma(n-3)]} = \frac{\Theta_k n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kj}\right]^2}{4[2+\gamma(n-3)]}. \quad (111)$$

The last equality in (111) reflects (86). ■

<u>In the presence of platform entry</u>.

If Pk enters Sj's market, (83) implies that the initial demand for Sj's product when Sj sells on Pk is:

$$q_{kj}^{S-E} = \frac{\left[d_j - p_{kj}^{S-E}\right][1+\gamma(n-2)] - \gamma\left[d_k - p_k^{P-E}\right] - \gamma \sum_{\substack{h=1\\ h\neq k\\ h\neq j}}^{n}\left(d_h - p_{kh}^{S-E}\right)}{[1-\gamma][1+\gamma(n-1)]}, \quad (112)$$

and the initial demand for Pk's product when Pk imitates category g product is:

$$q_k^{P-E} = \frac{\left[d_k - p_k^{P-E}\right][1+\gamma(n-2)] - \gamma \sum_{\substack{h=1\\ h\neq k}}^{n}\left(d_h - p_{kh}^{S-E}\right)}{[1-\gamma][1+\gamma(n-1)]}. \quad (113)$$

Define:

$$\Delta_{kj} \equiv \frac{\left[d_j - c_j^S\right]\left[1+\gamma\left(n-2\right)\right] - \gamma\left[d_k - c_k^P\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(d_h - c_h^S\right)}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]} \tag{114}$$

to be Sj's "selling strength" on Pk when Pk enters Sj's market, and

$$\overline{\Delta}_{kj} \equiv \frac{\left[d_k - c_k^P\right]\left[1+\gamma\left(n-2\right)\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n}\left(d_h - c_h^S\right)}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}. \tag{115}$$

to be Pk's "selling strength" when Pk competes with third-party sellers in seller market.

Assumption 2 and (114) imply that the selling strength of Sj in category g, when competing with Pk, is:

$$\Delta_{kjg} = \frac{\left[d^{Sg} - c^{Sg}\right]\left[1+\gamma\left(n-2\right)\right] - \gamma\left[d_k - c_k^P\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(d^{Sg} - c^{Sg}\right)}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}$$

$$= \frac{\left[d^{Sg} - c^{Sg}\right]\left[1+\gamma\left(n-2\right)\right] - \gamma\left[d_k - c_k^P\right] - \gamma\left[n-2\right]\left[d^{Sg} - c^{Sg}\right]}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}$$

$$= \frac{\left[d^{Sg} - c^{Sg}\right]\left[1+\gamma\left(n-2\right) - \gamma\left(n-2\right)\right] - \gamma\left[d_k - c_k^P\right]}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]} = \frac{d^{Sg} - c^{Sg} - \gamma\left[d_k - c_k^P\right]}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}. \tag{116}$$

Assumption 2 and (115) imply that the selling strength of Pk, when competing with Sj in category g, is:

$$\overline{\Delta}_{kjg} = \frac{\left[d_k - c_k^P\right]\left[1+\gamma\left(n-2\right)\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n}\left(d^{Sg} - c^{Sg}\right)}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}$$

$$= \frac{\left[d_k - c_k^P\right]\left[1+\gamma\left(n-2\right)\right] - \gamma\left[n-1\right]\left[d^{Sg} - c^{Sg}\right]}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}. \tag{117}$$

Assumptions 2 and 3, (116), and (117) imply that for $j \neq l$, $j, l \in \{1, 2, ..., n\}$, $k, g \in \{1, 2\}$:

$$\Delta_{kjg} = \Delta_{k\,\mathrm{lg}} = \Delta_{kg},\ \Delta_{k1} > \Delta_{k2},\ \Delta_{1g} < \Delta_{2g}; \tag{118}$$

$$\overline{\Delta}_{kjg} = \overline{\Delta}_{k\,\mathrm{lg}} = \overline{\Delta}_{kg},\ \overline{\Delta}_{k1} < \overline{\Delta}_{k2},\ \overline{\Delta}_{1g} > \overline{\Delta}_{2g}. \tag{119}$$

Lemma 22. *Suppose Assumption 2 holds. Further suppose Sj ($j \in \{1, 2, ..., n-1\}$) in category g ($g \in \{1, 2\}$) sells on Pk ($k \in \{1, 2\}$) and Pk enters Sj's market. Given w_{kg}^E, Sj's* *equilibrium output (i.e., sales)* (Q_{kj}^{S-E}) *is* $\frac{\Theta_k [1+\gamma(n-2)] \left\{ \Delta_{kj} 2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma - 2w_{kg}^E \right\}}{f(r,n)}$, *and Pk's equilibrium output (i.e., sales)* (q_k^{P-E}) *is* $\frac{[1+\gamma(n-2)] \left\{ \Delta_{kj}\gamma[n-1] + \overline{\Delta}_{kj}[2+\gamma(n-2)] \right\} - w_{kg}^E \gamma [n-1]}{f(r,n)}$.

Proof. (112) implies that Sj's profit is:

$$\pi_{kj}^{S-E} \;=\; \left[p_{kj}^{S-E} - w_{kg}^E - c_j^S \right] \Theta_k \, q_{kj}^{S-E}. \tag{120}$$

(113) implies that Pk's profit is:

$$\Pi_k^{P-E} \;=\; \left[p_k^{P-E} - c_k^P \right] \Theta_k \, q_k^{P-E} - F + w_{kg}^E \, \Theta_k \sum_{\substack{j=1 \\ j \neq k}}^{n} q_{kj}^{S-E} + \overline{\Pi}_{kl}, \tag{121}$$

where $\overline{\Pi}_{kl}$ is the profit that Pk secures from sellers in category l ($l \in \{1,2\}$, $l \neq g$), $\overline{\Pi}_{kl} > 0$ if sellers in category l sell on Pk, and $\overline{\Pi}_{kl} = 0$ if they do not sell on Pk.

(112), (113), and (121) imply that Pk chooses its price p_{kj}^P to maximize Π_k:

$$\frac{\partial \Pi_k^{P-E}}{\partial p_k^{P-E}} \;=\; 0 \;\;\Leftrightarrow\;\; q_k^{P-E} + \left[p_k^{P-E} - c_k^P \right] \frac{\partial q_k^{P-E}}{\partial p_k^{P-E}} + w_{kg}^E \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\partial q_{kj}^{S-E}}{\partial p_k^{P-E}} \;=\; 0$$

$$\Leftrightarrow \;\; \frac{\left[d_k - p_k^{P-E} \right] [1 + \gamma(n-2)] - \gamma \sum_{\substack{h=1 \\ h \neq k}}^{n} \left(d_h - p_{kh}^{S-E} \right)}{[1-\gamma][1+\gamma(n-1)]} - \frac{\left[p_k^{P-E} - c_k^P \right] [1+\gamma(n-2)]}{[1-\gamma][1+\gamma(n-1)]} + w_{kg}^E \frac{\gamma[n-1]}{[1-\gamma][1+\gamma(n-1)]} \;=\; 0$$

$$\Leftrightarrow \;\; \left[d_k - p_k^{P-E} \right] [1 + \gamma(n-2)] - \gamma \sum_{\substack{h=1 \\ h \neq k}}^{n} \left(d_h - p_{kh}^{S-E} \right) - \left[p_k^{P-E} - c_k^P \right] [1+\gamma(n-2)] + w_{kg}^E \gamma [n-1] \;=\; 0$$

$$\Leftrightarrow \;\; 2\, p_k^{P-E} [1+\gamma(n-2)] \;=\; \left[d_k + c_k^P \right] [1+\gamma(n-2)] - \gamma \sum_{\substack{h=1 \\ h \neq k}}^{n} \left(d_h - p_{kh}^{S-E} \right) + w_{kg}^E \gamma [n-1]$$

$$\Leftrightarrow \;\; p_k^{P-E} \;=\; \frac{d_k + c_k^P}{2} - \frac{\gamma \sum_{\substack{h=1 \\ h \neq k}}^{n} \left(d_h - p_{kh}^{S-E} \right)}{2[1+\gamma(n-2)]} + \frac{w_{kg}^E \gamma [n-1]}{2[1+\gamma(n-2)]}. \tag{122}$$

$$\Leftrightarrow \quad p_k^{P-E} = \frac{[\,1+\gamma\,(\,n-2\,)\,]\left[\,d_k + c_k^P\,\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,d_h - p_{kh}^{S-E}\,\right) + w_{kg}^E\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,]\left[\,d_k - c_k^P + 2c_k^P\,\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,d_h - c_h^S + c_h^S - p_{kh}^{S-E}\,\right) + w_{kg}^E\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}$$

$$= \frac{1}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\Bigg\{\,[\,1+\gamma\,(\,n-2\,)\,]\left[\,d_k - c_k^P\,\right] + 2c_k^P\,[\,1+\gamma\,(\,n-2\,)\,]$$

$$- \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,d_h - c_h^S\,\right) - \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,c_h^S - p_{kh}^{S-E}\,\right) + w_{kg}^E\,\gamma\,[\,n-1\,]\,\Bigg\}$$

$$= \frac{1}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\Bigg\{\,[\,1+\gamma\,(\,n-2\,)\,]\left[\,d_k - c_k^P\,\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,d_h - c_h^S\,\right)$$

$$+ 2c_k^P\,[\,1+\gamma\,(\,n-2\,)\,] + \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,p_{kh}^{S-E} - c_h^S\,\right) + w_{kg}^E\,\gamma\,[\,n-1\,]\,\Bigg\}. \quad (123)$$

(115) and (123) imply that:

$$p_k^{P-E} = \frac{\overline{\Delta}_{kj}\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,] + \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,p_{kh}^{S-E} - c_h^S\,\right) + w_{kg}^E\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} + c_k^P$$

$$\Leftrightarrow \quad p_k^{P-E} - c_k^P = \frac{\overline{\Delta}_{kj}\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,] + \gamma \sum\limits_{\substack{h=1\\h\neq k}}^{n} \left(\,p_{kh}^{S-E} - c_h^S\,\right) + w_{kg}^E\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \quad (124)$$

(112) and (120) imply that Sj chooses its price p_{kj}^S to maximize π_{kj}^{S-E}:

$$\frac{\partial \pi_{kj}^{S-E}}{\partial p_{kj}^{S-E}} = 0 \quad \Leftrightarrow \quad q_{kj}^{S-E} + \left[\,p_{kj}^{S-E} - w_{kg}^E - c_j^S\,\right] \frac{\partial q_{kj}^{S-E}}{\partial p_{kj}^{S-E}} = 0$$

$$\Leftrightarrow \quad \frac{\left[\,d_j - p_{kj}^{S-E}\,\right][\,1+\gamma\,(\,n-2\,)\,] - \gamma\left[\,d_k - p_k^{P-E}\,\right] - \gamma \sum\limits_{\substack{h=1\\h\neq k\\h\neq j}}^{n} \left(\,d_h - p_{kh}^{S-E}\,\right)}{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}$$

$$
- \frac{\left[\, p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \,\right] [\, 1 + \gamma\, (\, n-2\,)\,]}{[\, 1-\gamma\,]\, [\, 1+\gamma\, (\, n-1\,)\,]} \;=\; 0
$$

$$
\Leftrightarrow \quad \left[\, d_j - p_{kj}^{S-E} \,\right] [\, 1+\gamma\, (\, n-2\,)\,] - \gamma \left[\, d_k - p_k^{P-E} \,\right] - \gamma \sum_{\substack{h=1\\ h\neq k\\ h\neq j}}^{n} \left(\, d_h - p_{kh}^{S-E} \,\right)
$$

$$
= \left[\, p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \,\right] [\, 1+\gamma\, (\, n-2\,)\,] \tag{125}
$$

$$
\Leftrightarrow \quad \left[\, d_j + w_{kg}^{E} + c_j^{S} \,\right] [\, 1+\gamma\, (\, n-2\,)\,] - \gamma \left[\, d_k - p_k^{P-E} \,\right] - \gamma \sum_{\substack{h=1\\ h\neq k\\ h\neq j}}^{n} \left(\, d_h - p_{kh}^{S-E} \,\right)
$$

$$
= \; 2\, p_{kj}^{S-E}\, [\, 1+\gamma\, (\, n-2\,)\,]
$$

$$
\Leftrightarrow \quad p_{kj}^{S-E} \;=\; \frac{d_j + w_{kg}^{E} + c_j^{S}}{2} - \frac{\gamma}{2\, [\, 1+\gamma\, (\, n-2\,)\,]} \left[\, d_k - p_k^{P-E} + \sum_{\substack{h=1\\ h\neq k\\ h\neq j}}^{n} \left(\, d_h - p_{kh}^{S-E} \,\right) \right] \tag{126}
$$

$$
\Leftrightarrow \quad p_{kj}^{S-E} \;=\; \frac{[\, 1+\gamma\, (\, n-2\,)\,]\, \left[\, d_j + w_{kg}^{E} + c_j^{S} \,\right] - \gamma \left[\, d_k - p_k^{P-E} + \sum_{\substack{h=1\\ h\neq k\\ h\neq j}}^{n} \left(\, d_h - p_{kh}^{S-E} \,\right) \right]}{2\, [\, 1+\gamma\, (\, n-2\,)\,]}
$$

$$
= \frac{[\, 1+\gamma\, (\, n-2\,)\,]\, \left[\, d_j - c_j^{S} + w_{kg}^{E} + 2 c_j^{S} \,\right] - \gamma \left[\, d_k - p_k^{P-E} + \sum_{\substack{h=1\\ h\neq k\\ h\neq j}}^{n} \left(\, d_h - p_{kh}^{S-E} \,\right) \right]}{2\, [\, 1+\gamma\, (\, n-2\,)\,]}
$$

$$
= \frac{[1+\gamma\, (\, n-2\,)]\, \left[d_j - c_j^{S} \right] - \gamma \left[d_k - c_k^{P} + c_k^{P} - p_k^{P-E} + \sum_{\substack{h=1\\ h\neq k\\ h\neq j}}^{n} \left(\, d_h - c_h^{S} + c_h^{S} - p_{kh}^{S-E} \,\right) \right]}{2\, [\, 1+\gamma\, (\, n-2\,)\,]}
$$

$$
+ \frac{w_{kg}^{E} + 2 c_j^{S}}{2}
$$

$$
\begin{aligned}
&= \frac{1}{2\left[1+\gamma\left(n-2\right)\right]}\Bigg\{\left[1+\gamma\left(n-2\right)\right]\left[d_j-c_j^S\right]-\gamma\left[d_k-c_k^P+\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(d_h-c_h^S\right)\right] \\
&\qquad -\gamma\left[c_k^P-p_k^{P-E}+\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(c_h^S-p_{kh}^{S-E}\right)\right]\Bigg\}+\frac{w_{kg}^E}{2}+c_j^S \\
&= \frac{1}{2\left[1+\gamma\left(n-2\right)\right]}\Bigg\{\left[1+\gamma\left(n-2\right)\right]\left[d^S-c^S\right]-\gamma\left[d_k-c_k^P+\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(d^S-c^S\right)\right] \\
&\qquad +\gamma\left[p_k^{P-E}-c_k^P+\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(p_{kh}^{S-E}-c_h^S\right)\right]\Bigg\}+\frac{w_{kg}^E}{2}+c_j^S \qquad (127) \\
&= \frac{1}{2\left[1+\gamma\left(n-2\right)\right]}\Bigg\{\left[1+\gamma\left(n-2\right)\right]\left[d^S-c^S\right]-\gamma\left[d_k-c_k^P+\left(n-2\right)\left(d^S-c^S\right)\right] \\
&\qquad +\gamma\left[p_k^{P-E}-c_k^P+\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(p_{kh}^{S-E}-c_h^S\right)\right]\Bigg\}+\frac{w_{kg}^E}{2}+c_j^S \\
&= \frac{1}{2\left[1+\gamma\left(n-2\right)\right]}\Bigg\{d^S-c^S-\gamma\left[d_k-c_k^P\right] \\
&\qquad +\gamma\left[p_k^{P-E}-c_k^P+\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(p_{kh}^{S-E}-c_h^S\right)\right]\Bigg\}+\frac{w_{kg}^E}{2}+c_j^S. \qquad (128)
\end{aligned}
$$

(127) reflects Assumption 2.

(116) and (128) imply that:

$$
p_{kj}^{S-E} = \frac{\Delta_{kj}\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]+\gamma\left[p_k^{P-E}-c_k^P+\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left(p_{kh}^{S-E}-c_h^S\right)\right]}{2\left[1+\gamma\left(n-2\right)\right]}+\frac{w_{kg}^E}{2}+c_j^S
$$

$$\Leftrightarrow \quad p_{kj}^{S-E} - c_j^S$$

$$= \frac{\Delta_{kj}\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,] + \gamma \left[p_k^{P-E} - c_k^P + \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(p_{kh}^{S-E} - c_h^S \right) \right]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} + \frac{w_{kg}^E}{2} \tag{129}$$

Summing (129) for all third-party sellers provides:

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \left(p_{kj}^{S-E} - c_j^S \right)$$

$$= \frac{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \frac{\gamma\,[\,n-1\,]\,\left[\,p_k^{P-E} - c_k^P\,\right]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} + \frac{w_{kg}^E\,[\,n-1\,]}{2}$$

$$+ \frac{\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j \neq k}}^{n} \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(p_{kh}^{S-E} - c_h^S \right)$$

$$\Leftrightarrow \quad \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(p_{kj}^{S-E} - c_j^S \right)$$

$$= \frac{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \frac{\gamma\,[\,n-1\,]\,\left[\,p_k^{P-E} - c_k^P\,\right]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} + \frac{w_{kg}^E\,[\,n-1\,]}{2}$$

$$+ \frac{\gamma\,[\,n-2\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(p_{kj}^{S-E} - c_j^S \right)$$

$$\Leftrightarrow \quad \left\{ 1 - \frac{\gamma\,[\,n-2\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \right\} \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(p_{kj}^{S-E} - c_j^S \right)$$

$$= \frac{[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \frac{\gamma\,[\,n-1\,]\,\left[\,p_k^{P-E} - c_k^P\,\right]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} + \frac{w_{kg}^E\,[\,n-1\,]}{2}$$

$$\Leftrightarrow \quad \frac{2+\gamma\,[\,n-2\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(p_{kj}^{S-E} - c_j^S \right)$$

$$= \frac{[1-\gamma][1+\gamma(n-1)]}{2[1+\gamma(n-2)]} \sum_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \frac{\gamma[n-1]\left[p_k^{P-E} - c_k^P\right]}{2[1+\gamma(n-2)]} + \frac{w_{kg}^E [n-1]}{2}$$

$$\Leftrightarrow \quad \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(p_{kj}^{S-E} - c_j^S\right)$$

$$= \frac{[1-\gamma][1+\gamma(n-1)] \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \gamma[n-1]\left[p_k^{P-E} - c_k^P\right] + w_{kg}^E [n-1][1+\gamma(n-2)]}{2+\gamma[n-2]}. \tag{130}$$

(124) and (130) imply that:

$$p_k^{P-E} - c_k^P = \frac{\overline{\Delta}_{kj}[1-\gamma][1+\gamma(n-1)] + w_{kg}^E \gamma[n-1]}{2[1+\gamma(n-2)]} + \frac{\gamma}{2[1+\gamma(n-2)]} \sum_{\substack{h=1 \\ h \neq k}}^{n} \left(p_{kh}^{S-E} - c_h^S\right)$$

$$\Leftrightarrow \quad p_k^{P-E} - c_k^P = \frac{\overline{\Delta}_{kj}[1-\gamma][1+\gamma(n-1)] + w_{kg}^E \gamma[n-1]}{2[1+\gamma(n-2)]}$$

$$+\gamma \frac{[1-\gamma][1+\gamma(n-1)] \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \gamma[n-1]\left[p_k^{P-E} - c_k^P\right] + w_{kg}^E [n-1][1+\gamma(n-2)]}{2[1+\gamma(n-2)][2+\gamma(n-2)]}$$

$$\Leftrightarrow \quad \left\{1 - \frac{\gamma^2[n-1]}{2[1+\gamma(n-2)][2+\gamma(n-2)]}\right\}\left[p_k^{P-E} - c_k^P\right]$$

$$= \frac{\overline{\Delta}_{kj}[1-\gamma][1+\gamma(n-1)] + w_{kg}^E \gamma[n-1]}{2[1+\gamma(n-2)]}$$

$$+ \frac{\gamma[1-\gamma][1+\gamma(n-1)] \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \gamma w_{kg}^E [n-1][1+\gamma(n-2)]}{2[1+\gamma(n-2)][2+\gamma(n-2)]}$$

$$\Leftrightarrow \quad \frac{2[1+\gamma(n-2)][2+\gamma(n-2)] - \gamma^2[n-1]}{2[1+\gamma(n-2)][2+\gamma(n-2)]} \left[p_k^{P-E} - c_k^P\right]$$

$$= \frac{\overline{\Delta}_{kj}[1-\gamma][1+\gamma(n-1)] + w_{kg}^E \gamma[n-1]}{2[1+\gamma(n-2)]}$$

$$+ \frac{\gamma[1-\gamma][1+\gamma(n-1)] \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} \Delta_{kj} + \gamma w_{kg}^E [n-1][1+\gamma(n-2)]}{2[1+\gamma(n-2)][2+\gamma(n-2)]}$$

$$\Leftrightarrow \quad f(r,n)\left[p_k^{P-E} - c_k^P\right]$$

$$= \overline{\Delta}_{kj}\,[1-\gamma]\,[1+\gamma\,(n-1)]\,[2+\gamma\,(n-2)] + w_{kg}^{E}\,\gamma\,[n-1]\,[2+\gamma\,(n-2)]$$

$$+\gamma\,[1-\gamma]\,[1+\gamma\,(n-1)]\sum_{\substack{j=1\\ j\neq k}}^{n}\Delta_{kj} + \gamma w_{kg}^{E}\,[n-1]\,[1+\gamma\,(n-2)] \tag{131}$$

$$\Leftrightarrow \quad f\,(r,n)\left[p_k^{P-E} - c_k^{P}\right] \;=\; \overline{\Delta}_{kj}\,[1-\gamma]\,[1+\gamma\,(n-1)]\,[2+\gamma\,(n-2)]$$

$$+\gamma\,[1-\gamma]\,[1+\gamma\,(n-1)]\sum_{\substack{j=1\\ j\neq k}}^{n}\Delta_{kj} + w_{kg}^{E}\,\gamma\,[n-1]\,[2+\gamma\,(n-2)+1+\gamma\,(n-2)]$$

$$\Leftrightarrow \quad f\,(r,n)\left[p_k^{P-E} - c_k^{P}\right] \;=\; [1-\gamma]\,[1+\gamma\,(n-1)]\left\{[2+\gamma\,(n-2)]\,\overline{\Delta}_{kj} + \gamma\sum_{\substack{j=1\\ j\neq k}}^{n}\Delta_{kj}\right\}$$

$$+\gamma\,[n-1]\,[3+2\gamma\,(n-2)]\,w_{kg}^{E}$$

$$\Leftrightarrow \quad p_k^{P-E} - c_k^{P}$$

$$= \frac{[1-\gamma]\,[1+\gamma\,(n-1)]\left\{[2+\gamma\,(n-2)]\,\overline{\Delta}_{kj} + \gamma\sum_{\substack{j=1\\ j\neq k}}^{n}\Delta_{kj}\right\} + \gamma\,[n-1]\,[3+2\gamma\,(n-2)]\,w_{kg}^{E}}{f\,(r,n)}$$

$$\Leftrightarrow \quad p_k^{P-E} - c_k^{P}$$

$$= \frac{[1-\gamma]\,[1+\gamma\,(n-1)]\left\{[2+\gamma\,(n-2)]\,\overline{\Delta}_{kj} + \gamma\,[n-1]\,\Delta_{kj}\right\} + \gamma\,[n-1]\,[3+2\gamma\,(n-2)]\,w_{kg}^{E}}{f\,(r,n)}. \tag{132}$$

(131) reflects (5). (132) reflcets $\Delta_{kj} = \Delta_{kh}$ for $\forall j, h \in \{1, 2, ..., n\}$, $j \neq k$, and $h \neq k$ from (116).

(125) implies that:

$$\left[d_j - p_{kj}^{S-E}\right][1+\gamma\,(n-2)] - \gamma\left[d_k - p_k^{P-E}\right] - \gamma\sum_{\substack{h=1\\ h\neq k}}^{n}\left(d_h - p_{kh}^{S-E}\right) + \gamma\left[d_j - p_{kj}^{S-E}\right]$$

$$= \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S}\right][1+\gamma\,(n-2)]$$

$$\Leftrightarrow \quad \left[d_j - p_{kj}^{S-E}\right][1+\gamma\,(n-2)+\gamma] - \gamma\left[d_k - p_k^{P-E}\right] - \gamma\sum_{\substack{h=1\\ h\neq k}}^{n}\left(d_h - p_{kh}^{S-E}\right)$$

$$= \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S}\right][1+\gamma\,(n-2)]$$

$$\Leftrightarrow \quad \left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 1 \right) \right] - \gamma \left[d_k - p_k^{P-E} \right] - \gamma \sum_{\substack{h=1 \\ h \neq k}}^{n} \left(d_h - p_{kh}^{S-E} \right)$$
$$= \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \right] \left[1 + \gamma \left(n - 2 \right) \right]$$

$$\Leftrightarrow \quad \left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 1 \right) \right] - \gamma \left[d_k - p_k^{P-E} \right] - \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \right] \left[1 + \gamma \left(n - 2 \right) \right]$$
$$= \gamma \sum_{\substack{h=1 \\ h \neq k}}^{n} \left(d_h - p_{kh}^{S-E} \right) \tag{133}$$

(122) and (133) imply that:

$$p_k^{P-E} = \frac{d_k + c_k^P}{2} + \frac{w_{kg}^{E} \gamma \left[n - 1 \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$
$$- \frac{\left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 1 \right) \right] - \gamma \left[d_k - p_k^{P-E} \right] - \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \right] \left[1 + \gamma \left(n - 2 \right) \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad p_k^{P-E} = \frac{d_k + c_k^P}{2} + \frac{w_{kg}^{E} \gamma \left[n - 1 \right] - w_{kg}^{E} \left[1 + \gamma \left(n - 2 \right) \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$
$$+ \frac{- \left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 1 \right) \right] + \gamma \left[d_k - p_k^{P-E} \right] + \left[p_{kj}^{S-E} - c_j^{S} \right] \left[1 + \gamma \left(n - 2 \right) \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad p_k^{P-E} = \frac{d_k + c_k^P}{2} + \frac{w_{kg}^{E} \left\{ \gamma \left[n - 1 \right] - \left[1 + \gamma \left(n - 2 \right) \right] \right\}}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$
$$+ \frac{- \left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 1 \right) \right] + \gamma \left[d_k - p_k^{P-E} \right] + \left[p_{kj}^{S-E} - d_j + d_j - c_j^{S} \right] \left[1 + \gamma \left(n - 2 \right) \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad p_k^{P-E} = \frac{d_k + c_k^P}{2} + \frac{d_j - c_j^S}{2} + \frac{w_{kg}^{E} \left[\gamma - 1 \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$
$$+ \frac{- \left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 1 \right) \right] + \gamma \left[d_k - p_k^{P-E} \right] + \left[p_{kj}^{S-E} - d_j \right] \left[1 + \gamma \left(n - 2 \right) \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad p_k^{P-E} = \frac{d_k + c_k^P}{2} + \frac{d_j - c_j^S}{2} + \frac{w_{kg}^{E} \left[\gamma - 1 \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$
$$+ \frac{- \left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 1 \right) \right] + \gamma \left[d_k - p_k^{P-E} \right] - \left[d_j - p_{kj}^{S-E} \right] \left[1 + \gamma \left(n - 2 \right) \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad p_k^{P-E} - d_k + d_k = \frac{d_k + c_k^P}{2} + \frac{d_j - c_j^S}{2}$$
$$+ \frac{w_{kg}^{E} \left[\gamma - 1 \right] - \left[d_j - p_{kj}^{S-E} \right] \left[2 + \gamma \left(2n - 3 \right) \right] + \gamma \left[d_k - p_k^{P-E} \right]}{2 \left[1 + \gamma \left(n - 2 \right) \right]}$$

$$\Leftrightarrow \quad p_k^{P-E} - d_k = \frac{-d_k + c_k^P}{2} + \frac{d_j - c_j^S}{2} + \frac{w_{kg}^E\,[\gamma - 1] - \left[d_j - p_{kj}^{S-E}\right][2 + \gamma(2n-3)] + \gamma\left[d_k - p_k^{P-E}\right]}{2[1 + \gamma(n-2)]}$$

$$\Leftrightarrow \quad 0 = \frac{d_j - c_j^S}{2} - \frac{d_k - c_k^P}{2} + d_k - p_k^{P-E} + \frac{w_{kg}^E\,[\gamma - 1] - \left[d_j - p_{kj}^{S-E}\right][2 + \gamma(2n-3)] + \gamma\left[d_k - p_k^{P-E}\right]}{2[1 + \gamma(n-2)]}$$

$$\Leftrightarrow \quad 0 = \frac{d_j - c_j^S}{2} - \frac{d_k - c_k^P}{2} + d_k - p_k^{P-E} + \frac{w_{kg}^E\,[\gamma - 1] - \left[d_j - c_j^S + c_j^S - p_{kj}^{S-E}\right][2 + \gamma(2n-3)] + \gamma\left[d_k - p_k^{P-E}\right]}{2[1 + \gamma(n-2)]}$$

$$\Leftrightarrow \quad 0 = \frac{d^S - c^S}{2} - \frac{\left[d^S - c^S\right][2 + \gamma(2n-3)]}{2[1 + \gamma(n-2)]} - \frac{d_k - c_k^P}{2} + d_k - p_k^{P-E} + \frac{w_{kg}^E\,[\gamma - 1] + \left[p_{kj}^{S-E} - c_j^S\right][2 + \gamma(2n-3)] + \gamma\left[d_k - p_k^{P-E}\right]}{2[1 + \gamma(n-2)]} \tag{134}$$

$$\Leftrightarrow \quad \frac{\left[p_{kj}^{S-E} - c_j^S\right][2 + \gamma(2n-3)]}{2[1 + \gamma(n-2)]} = -\frac{d^S - c^S}{2} + \frac{\left[d^S - c^S\right][2 + \gamma(2n-3)]}{2[1 + \gamma(n-2)]} + \frac{d_k - c_k^P}{2} - \left[d_k - p_k^{P-E}\right] - \frac{w_{kg}^E\,[\gamma - 1] + \gamma\left[d_k - p_k^{P-E}\right]}{2[1 + \gamma(n-2)]}. \tag{135}$$

(134) reflects Assumption 2. (135) implies $\forall j, h \in \{1, 2, ..., n\}$, $j \neq k$, and $h \neq k$:

$$p_{kj}^{S-E} - c_j^S = p_{kh}^{S-E} - c_h^S. \tag{136}$$

(125) implies that:

$$\left[d_j - c_j^S + c_j^S - p_{kj}^{S-E}\right][1 + \gamma(n-2)] - \gamma\left[d_k - c_k^P + c_k^P - p_k^{P-E}\right] - \gamma \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(d_h - c_h^S + c_h^S - p_{kh}^{S-E}\right) = \left[p_{kj}^{S-E} - w_{kg}^E - c_j^S\right][1 + \gamma(n-2)]$$

$$\Leftrightarrow \quad \left[d_j - c_j^S\right][1 + \gamma(n-2)] - \left[p_{kj}^{S-E} - c_j^S\right][1 + \gamma(n-2)] - \gamma\left[d_k - c_k^P\right] + \gamma\left[p_k^{P-E} - c_k^P\right] - \gamma \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(d_h - c_h^S\right) + \gamma \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left[p_{kh}^{S-E} - c_h^S\right] = \left[p_{kj}^{S-E} - w_{kg}^E - c_j^S\right][1 + \gamma(n-2)]$$

$$\Leftrightarrow \quad \left[d^S - c^S\right][1 + \gamma(n-2)] - \left[p_{kj}^{S-E} - c_j^S\right][1 + \gamma(n-2)] - \gamma\left[d_k - c_k^P\right] + \gamma\left[p_k^{P-E} - c_k^P\right]$$

$$-\gamma\,[\,n-2\,]\,\left[\,d^S-c^S\,\right]+\gamma\sum_{\substack{h=1\\h\neq k\\h\neq j}}^{n}\left[p_{kh}^{S-E}-c_h^S\right]=\ \left[\,p_{kj}^{S-E}-c_j^S\,\right][\,1+\gamma\,(\,n-2\,)\,]-w_{kg}^E\,[\,1+\gamma\,(\,n-2\,)\,] \tag{137}$$

$$\Leftrightarrow\quad \left[\,d^S-c^S\,\right][\,1+\gamma\,(\,n-2\,)\,]-\gamma\,[\,n-2\,]\left[\,d^S-c^S\,\right]-\gamma\left[\,d_k-c_k^P\right]-2\left[p_{kj}^{S-E}-c_j^S\right][\,1+\gamma\,(\,n-2\,)\,]$$
$$+\gamma\left[p_k^{P-E}-c_k^P\right]+\gamma\,[\,n-2\,]\left[p_{kj}^{S-E}-c_j^S\right]\ =\ -\,w_{kg}^E\,[\,1+\gamma\,(\,n-2\,)\,] \tag{138}$$

$$\Leftrightarrow\quad d^S-c^S-\gamma\left[\,d_k-c_k^P\right]-\left[p_{kj}^{S-E}-c_j^S\right][\,2+2\gamma\,(\,n-2\,)-\gamma\,[\,n-2\,]\,]+\gamma\left[p_k^{P-E}-c_k^P\right]$$
$$=\ -\,w_{kg}^E\,[\,1+\gamma\,(\,n-2\,)\,]$$

$$\Leftrightarrow\quad d^S-c^S-\gamma\left[\,d_k-c_k^P\right]-\left[p_{kj}^{S-E}-c_j^S\right][\,2+\gamma\,[\,n-2\,]\,]+\gamma\left[p_k^{P-E}-c_k^P\right]$$
$$=\ -\,w_{kg}^E\,[\,1+\gamma\,(\,n-2\,)\,] \tag{139}$$

(137) relfects Assumption 2. (138) reflects (136).

(124) and (136) imply that:

$$p_k^{P-E}-c_k^P\ =\ \frac{\overline{\Delta}_{kj}\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]+\gamma\,[\,n-1\,]\left(p_{kh}^{S-E}-c_h^S\right)+w_{kg}^E\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]} \tag{140}$$

(139) and (140) imply that:

$$d^S-c^S-\gamma\left[\,d_k-c_k^P\right]-\left[p_{kj}^{S-E}-c_j^S\right][\,2+\gamma\,(\,n-2\,)\,]+w_{kg}^E\,[\,1+\gamma\,(\,n-2\,)\,]$$
$$+\gamma\frac{\overline{\Delta}_{kj}\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]+\gamma\,[\,n-1\,]\left(p_{kh}^{S-E}-c_h^S\right)+w_{kg}^E\,\gamma\,[\,n-1\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\ =\ 0$$

$$\Leftrightarrow \Delta_{kj}\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]+\frac{\overline{\Delta}_{kj}\gamma\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}+w_{kg}^E\left[\frac{\gamma^2\,(\,n-1\,)}{2\,[\,1+\gamma\,(\,n-2\,)\,]}+\,1+\gamma\,(\,n-2\,)\right]$$
$$=\left[p_{kj}^{S-E}-c_j^S\right]\left[2+\gamma\,(\,n-2\,)-\frac{\gamma^2\,(\,n-1\,)}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\right] \tag{141}$$

$$\Leftrightarrow [\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,\frac{\Delta_{kj}2\,[\,1+\gamma\,(\,n-2\,)\,]+\overline{\Delta}_{kj}\gamma}{2\,[\,1+\gamma\,(\,n-2\,)\,]}$$
$$+w_{kg}^E\left[\frac{\gamma^2\,(\,n-1\,)+2\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,]}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\right]$$
$$=\left[p_{kj}^{S-E}-c_j^S\right]\left[\frac{2\,[\,1+\gamma\,(\,n-2\,)\,]\,[2+\gamma\,(\,n-2\,)]-\gamma^2\,(\,n-1\,)}{2\,[\,1+\gamma\,(\,n-2\,)\,]}\right]$$

$$\Leftrightarrow [\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\left\{\Delta_{kj}2\,[\,1+\gamma\,(\,n-2\,)\,]+\overline{\Delta}_{kj}\gamma\right\}$$
$$+w_{kg}^E\left\{\gamma^2\,(\,n-1\,)+2\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,1+\gamma\,(\,n-2\,)\,]\right\}$$

$$= \left[p_{kj}^{S-E} - c_j^S\right] \left\{ 2\left[1+\gamma\left(n-2\right)\right]\left[2+\gamma\left(n-2\right)\right] - \gamma^2\left(n-1\right)\right\}$$

$$\begin{aligned} \Leftrightarrow & \left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]\left\{\Delta_{kj} 2\left[1+\gamma\left(n-2\right)\right] + \overline{\Delta}_{kj}\gamma\right\} \\ & \quad + w_{kg}^E \left\{\gamma^2\left(n-1\right) + 2\left[1+\gamma\left(n-2\right)\right]\left[1+\gamma\left(n-2\right)\right]\right\} \\ & = \left[p_{kj}^{S-E} - c_j^S\right] f\left(r,n\right) \end{aligned} \tag{142}$$

$$\begin{aligned} \Leftrightarrow p_{kj}^{S-E} - c_j^S = \frac{1}{f\left(r,n\right)} \Bigg\{ & \left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]\left\{\Delta_{kj} 2\left[1+\gamma\left(n-2\right)\right] + \overline{\Delta}_{kj}\gamma\right\} \\ & + w_{kg}^E \left\{\gamma^2\left(n-1\right) + 2\left[1+\gamma\left(n-2\right)\right]\left[1+\gamma\left(n-2\right)\right]\right\} \Bigg\}. \end{aligned} \tag{143}$$

(141) reflects (116). (142) reflects (5).

(126) implies that:

$$d_j - p_{kj}^{S-E} = \frac{d_j - w_{kg}^E - c_j^S}{2} + \frac{\gamma}{2\left[1+\gamma\left(n-2\right)\right]} \left[d_k - p_k^{P-E} + \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-E}\right) \right]. \tag{144}$$

(144) implies that:

$$\begin{aligned} & \gamma \left[d_k - p_k^{P-E} + \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-E}\right) \right] \\ = \; & 2\left[1+\gamma\left(n-2\right)\right]\left[d_j - p_{kj}^{S-E}\right] - \left[1+\gamma\left(n-2\right)\right]\left[d_j - w_{kg}^E - c_j^S\right]. \end{aligned} \tag{145}$$

(112) and (145) imply that:

$$q_{kj}^{S-E} = \frac{\left[d_j - p_{kj}^{S-E}\right]\left[1+\gamma\left(n-2\right)\right] - \gamma \left[d_k - p_k^{P-E} + \sum_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-E}\right) \right]}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}$$

$$= \frac{\left[d_j - p_{kj}^{S-E}\right]\left[1+\gamma\left(n-2\right)\right] - \left\{2\left[1+\gamma\left(n-2\right)\right]\left[d_j - p_{kj}^{S-E}\right] - \left[1+\gamma\left(n-2\right)\right]\left[d_j - w_{kg}^E - c_j^S\right]\right\}}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}$$

$$= \frac{\left[d_j - p_{kj}^{S-E}\right]\left[1+\gamma\left(n-2\right)\right] - 2\left[1+\gamma\left(n-2\right)\right]\left[d_j - p_{kj}^{S-E}\right] + \left[1+\gamma\left(n-2\right)\right]\left[d_j - w_{kg}^E - c_j^S\right]}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}$$

$$= \frac{-\left[1+\gamma\left(n-2\right)\right]\left[d_j - p_{kj}^{S-E}\right] + \left[1+\gamma\left(n-2\right)\right]\left[d_j - w_{kg}^E - c_j^S\right]}{\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]}$$

$$= \frac{[1+\gamma(n-2)]\left[d_j - w_{kg}^E - c_j^S - \left(d_j - p_{kj}^{S-E}\right)\right]}{[1-\gamma][1+\gamma(n-1)]}$$

$$= \frac{[1+\gamma(n-2)]\left[p_{kj}^{S-E} - w_{kg}^E - c_j^S\right]}{[1-\gamma][1+\gamma(n-1)]}. \tag{146}$$

(143) and (146) imply that:

$$q_{kj}^{S-E} = \frac{[1+\gamma(n-2)]}{[1-\gamma][1+\gamma(n-1)]}$$

$$\cdot \frac{1}{f(r,n)} \Bigg\{ [1-\gamma][1+\gamma(n-1)]\left\{\Delta_{kj} 2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}$$

$$+ w_{kg}^E \left\{\gamma^2(n-1) + 2[1+\gamma(n-2)][1+\gamma(n-2)] - f(r,n)\right\} \Bigg\}$$

$$= \frac{[1+\gamma(n-2)]}{f(r,n)[1-\gamma][1+\gamma(n-1)]}$$

$$\cdot \Bigg\{ [1-\gamma][1+\gamma(n-1)]\left\{\Delta_{kj} 2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}$$

$$+ w_{kg}^E \left\{2\gamma^2(n-1) + 2[1+\gamma(n-2)][1+\gamma(n-2) - (2+\gamma[n-2])]\right\} \Bigg\} \tag{147}$$

$$= \frac{[1+\gamma(n-2)]}{f(r,n)[1-\gamma][1+\gamma(n-1)]}$$

$$\cdot \Bigg\{ [1-\gamma][1+\gamma(n-1)]\left\{\Delta_{kj} 2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}$$

$$+ w_{kg}^E \left\{2\gamma^2(n-1) - 2[1+\gamma(n-2)]\right\} \Bigg\}$$

$$= \frac{[1+\gamma(n-2)]\left\{\Delta_{kj} 2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}}{f(r,n)} + \frac{[1+\gamma(n-2)]\, 2w_{kg}^E \left\{\gamma^2(n-1) - [1+\gamma(n-2)]\right\}}{f(r,n)[1-\gamma][1+\gamma(n-1)]}$$

$$= \frac{[1+\gamma(n-2)]\left\{\Delta_{kj} 2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}}{f(r,n)} - \frac{[1+\gamma(n-2)]\, 2w_{kg}^E}{f(r,n)} \tag{148}$$

$$= \frac{[1+\gamma(n-2)]\left\{\Delta_{kj} 2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma - 2w_{kg}^E\right\}}{f(r,n)}. \tag{149}$$

(147) reflects (5). (148) reflects that $[1-\gamma][1+\gamma(n-1)] = 1-\gamma^2(n-1)+\gamma(n-2) = -\left\{\gamma^2(n-1) - [1+\gamma(n-2)]\right\}$.

(120) and (146) imply that Sj's profit is:

$$\pi_{kj}^{S-E} = \frac{\Theta_k [1-\gamma][1+\gamma(n-1)]}{[1+\gamma(n-2)]} \left[q_{kj}^{S-E} \right]^2 . \tag{150}$$

(113) implies that:

$$\begin{aligned}
q_k^{P-E} &= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-2)] - \gamma \sum\limits_{\substack{h=1 \\ h \neq k}}^{n} \left(d_h - p_{kh}^{S-E} \right)}{[1-\gamma][1+\gamma(n-1)]} \\
&= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-2)+\gamma] - \gamma \left[d_k - p_k^{P-E} \right] - \gamma \sum\limits_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-E} \right) - \gamma \left[d_j - p_{kj}^{S-E} \right]}{[1-\gamma][1+\gamma(n-1)]} \\
&= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - \gamma \left[d_j - p_{kj}^{S-E} \right] - \gamma \left[d_k - p_k^{P-E} + \sum\limits_{\substack{h=1 \\ h \neq k \\ h \neq j}}^{n} \left(d_h - p_{kh}^{S-E} \right) \right]}{[1-\gamma][1+\gamma(n-1)]} \\
&= \frac{1}{[1-\gamma][1+\gamma(n-1)]} \Big\{ \left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - \gamma \left[d_j - p_{kj}^{S-E} \right] \\
&\qquad -2 [1+\gamma(n-2)] \left[d_j - p_{kj}^{S-E} \right] + [1+\gamma(n-2)] \left[d_j - w_{kg}^{E} - c_j^{S} \right] \Big\} \qquad (151) \\
&= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - \gamma \left[d_j - p_{kj}^{S-E} \right] + [1+\gamma(n-2)] \left[d_j - w_{kg}^{E} - c_j^{S} - 2 \left(d_j - p_{kj}^{S-E} \right) \right]}{[1-\gamma][1+\gamma(n-1)]} \\
&= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - \gamma \left[d_j - p_{kj}^{S-E} \right] + [1+\gamma(n-2)] \left[2 p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} - d_j \right]}{[1-\gamma][1+\gamma(n-1)]} \\
&= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - \gamma \left[d_j - p_{kj}^{S-E} \right] + [1+\gamma(n-2)] \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} + p_{kj}^{S-E} - d_j \right]}{[1-\gamma][1+\gamma(n-1)]} \\
&= \frac{1}{[1-\gamma][1+\gamma(n-1)]} \Big\{ \left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - \gamma \left[d_j - p_{kj}^{S-E} \right] \\
&\qquad - [1+\gamma(n-2)] \left[d_j - p_{kj}^{S-E} \right] + [1+\gamma(n-2)] \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \right] \Big\} \\
&= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - [1+\gamma(n-2+1)] \left[d_j - p_{kj}^{S-E} \right] + [1+\gamma(n-2)] \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \right]}{[1-\gamma][1+\gamma(n-1)]} \\
&= \frac{\left[d_k - p_k^{P-E} \right] [1+\gamma(n-1)] - [1+\gamma(n-1)] \left[d_j - p_{kj}^{S-E} \right] + [1+\gamma(n-2)] \left[p_{kj}^{S-E} - w_{kg}^{E} - c_j^{S} \right]}{[1-\gamma][1+\gamma(n-1)]}
\end{aligned}$$

$$= \frac{[1+\gamma(n-1)]\left[d_k - p_k^{P-E} - \left(d_j - p_{kj}^{S-E}\right)\right] + [1+\gamma(n-2)]\left[p_{kj}^{S-E} - w_{kg}^E - c_j^S\right]}{[1-\gamma][1+\gamma(n-1)]}$$

$$= \frac{d_k - p_k^{P-E} - \left[d_j - p_{kj}^{S-E}\right]}{1-\gamma} + \frac{[1+\gamma(n-2)]\left[p_{kj}^{S-E} - w_{kg}^E - c_j^S\right]}{[1-\gamma][1+\gamma(n-1)]}$$

$$= \frac{d_k - p_k^{P-E} - \left[d_j - p_{kj}^{S-E}\right]}{1-\gamma} + q_{kj}^{S-E}. \tag{152}$$

(152) reflects (146) and (151) reflects (145).

Assumption 2, (132), (143), (149), and (152) imply that:

$$q_k^{P-E} = \frac{d_k - c_k^P - \left[p_k^{P-E} - c_k^P\right] - \left[d_j - c_j^S - \left(p_{kj}^{S-E} - c_j^S\right)\right]}{1-\gamma} + q_{kj}^{S-E}$$

$$= \frac{d_k - c_k^P - \left[d^S - c^S\right]}{1-\gamma}$$

$$- \frac{[1-\gamma][1+\gamma(n-1)]\left\{[2+\gamma(n-2)]\overline{\Delta}_{kj} + \gamma[n-1]\Delta_{kj}\right\} + \gamma[n-1][3+2\gamma(n-2)]w_{kg}^E}{[1-\gamma]f(r,n)}$$

$$+ \frac{1}{[1-\gamma]f(r,n)}\Big\{[1-\gamma][1+\gamma(n-1)]\left\{\Delta_{kj}2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}$$

$$+ w_{kg}^E\left\{\gamma^2(n-1) + 2[1+\gamma(n-2)][1+\gamma(n-2)]\right\}\Big\}$$

$$+ \frac{[1+\gamma(n-2)]\left\{\Delta_{kj}2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma - 2w_{kg}^E\right\}}{f(r,n)}$$

$$= \frac{d_k - c_k^P - \left[d^S - c^S\right]}{1-\gamma}$$

$$- \frac{[1-\gamma][1+\gamma(n-1)]\left\{[2+\gamma(n-2)]\overline{\Delta}_{kj} + \gamma[n-1]\Delta_{kj}\right\}}{[1-\gamma]f(r,n)}$$

$$- \frac{\gamma[n-1][3+2\gamma(n-2)]w_{kg}^E}{[1-\gamma]f(r,n)}$$

$$+ \frac{[1-\gamma][1+\gamma(n-1)]\left\{\Delta_{kj}2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}}{[1-\gamma]f(r,n)}$$

$$+ \frac{w_{kg}^E\left\{\gamma^2(n-1) + 2[1+\gamma(n-2)][1+\gamma(n-2)]\right\}}{[1-\gamma]f(r,n)}$$

$$+ \frac{[1+\gamma(n-2)]\left\{\Delta_{kj}2[1+\gamma(n-2)] + \overline{\Delta}_{kj}\gamma\right\}}{f(r,n)} - \frac{2w_{kg}^E[1+\gamma(n-2)]}{f(r,n)}$$

$$= \frac{d_k - c_k^P - \left[d^S - c^S\right]}{1-\gamma}$$

$$+\frac{[1-\gamma][1+\gamma(n-1)]\left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma-[2+\gamma(n-2)]\overline{\Delta}_{kj}-\gamma[n-1]\Delta_{kj}\right\}}{[1-\gamma]f(r,n)}$$
$$+\frac{w_{kg}^{E}\left\{\gamma^{2}(n-1)+2[1+\gamma(n-2)]^{2}-\gamma[n-1][3+2\gamma(n-2)]-2[1-\gamma][1+\gamma(n-2)]\right\}}{[1-\gamma]f(r,n)}$$
$$+\frac{[1+\gamma(n-2)]\left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma\right\}}{f(r,n)}. \tag{153}$$

Observe that:

$$\frac{[1-\gamma][1+\gamma(n-1)]\left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma-[2+\gamma(n-2)]\overline{\Delta}_{kj}-\gamma[n-1]\Delta_{kj}\right\}}{[1-\gamma]f(r,n)}$$
$$=\frac{[1+\gamma(n-1)]\left\{\Delta_{kj}[2+2\gamma(n-2)-\gamma(n-1)]-[2+\gamma(n-2)-\gamma]\overline{\Delta}_{kj}\right\}}{f(r,n)}$$
$$=\frac{[1+\gamma(n-1)]\left\{\Delta_{kj}[2+\gamma(n-3)]-[2+\gamma(n-3)]\overline{\Delta}_{kj}\right\}}{f(r,n)}$$
$$=\frac{[1+\gamma(n-1)][2+\gamma(n-3)]\left[\Delta_{kj}-\overline{\Delta}_{kj}\right]}{f(r,n)}; \tag{154}$$

$$\Delta_{kj}-\overline{\Delta}_{kj}=\frac{d^{S}-c^{S}-\gamma\left[d_{k}-c_{k}^{P}\right]-\left[d_{k}-c_{k}^{P}\right][1+\gamma(n-2)]+\gamma[n-1]\left[d^{S}-c^{S}\right]}{[1-\gamma][1+\gamma(n-1)]} \tag{155}$$
$$=\frac{\left[d^{S}-c^{S}\right][1+\gamma(n-1)]-\left[d_{k}-c_{k}^{P}\right][1+\gamma(n-2)+\gamma]}{[1-\gamma][1+\gamma(n-1)]}$$
$$=\frac{\left[d^{S}-c^{S}\right][1+\gamma(n-1)]-\left[d_{k}-c_{k}^{P}\right][1+\gamma(n-1)]}{[1-\gamma][1+\gamma(n-1)]}=\frac{d^{S}-c^{S}-\left[d_{k}-c_{k}^{P}\right]}{[1-\gamma]}; \tag{156}$$

$$\gamma^{2}(n-1)+2[1+\gamma(n-2)]^{2}-\gamma[n-1][3+2\gamma(n-2)]-2[1-\gamma][1+\gamma(n-2)]$$
$$=\gamma^{2}(n-1)+2[1+\gamma(n-2)][1+\gamma(n-2)-(1-\gamma)]-\gamma[n-1][3+2\gamma(n-2)]$$
$$=\gamma^{2}(n-1)+2[1+\gamma(n-2)][\gamma(n-2)+\gamma]-\gamma[n-1][3+2\gamma(n-2)]$$
$$=\gamma^{2}(n-1)+2\gamma[1+\gamma(n-2)][n-1]-\gamma[n-1][3+2\gamma(n-2)]$$
$$=\gamma[n-1]\{\gamma+2[1+\gamma(n-2)]-[3+2\gamma(n-2)]\}$$
$$=\gamma[n-1][\gamma+2+2\gamma(n-2)-3-2\gamma(n-2)]=\gamma[n-1][\gamma-1]. \tag{157}$$

(155) reflects (116) and (117).

(153), (154), (156), and (157) imply that:

$$q_{k}^{P-E}=\overline{\Delta}_{kj}-\Delta_{kj}+\frac{[1+\gamma(n-1)][2+\gamma(n-3)]\left[\Delta_{kj}-\overline{\Delta}_{kj}\right]}{f(r,n)}+\frac{w_{kg}^{E}\gamma[n-1][\gamma-1]}{[1-\gamma]f(r,n)}$$

$$+\frac{[1+\gamma(n-2)]\left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma\right\}}{f(r,n)}$$

$$=\frac{\left[\Delta_{kj}-\overline{\Delta}_{kj}\right]\{[1+\gamma(n-1)][2+\gamma(n-3)]-f(r,n)\}}{f(r,n)}-\frac{w_{kg}^{E}\gamma[n-1]}{f(r,n)}$$

$$+\frac{[1+\gamma(n-2)]\left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma\right\}}{f(r,n)}$$

$$=\frac{\Delta_{kj}\{[1+\gamma(n-1)][2+\gamma(n-3)]-f(r,n)\}}{f(r,n)}-\frac{w_{kg}^{E}\gamma[n-1]}{f(r,n)}$$

$$-\frac{\overline{\Delta}_{kj}\{[1+\gamma(n-1)][2+\gamma(n-3)]-f(r,n)\}}{f(r,n)}$$

$$+\frac{2\Delta_{kj}[1+\gamma(n-2)][1+\gamma(n-2)]}{f(r,n)}+\frac{\overline{\Delta}_{kj}\gamma[1+\gamma(n-2)]}{f(r,n)}$$

$$=\frac{\Delta_{kj}\{[1+\gamma(n-1)][2+\gamma(n-3)]-f(r,n)+2[1+\gamma(n-2)][1+\gamma(n-2)]\}}{f(r,n)}$$

$$-\frac{\overline{\Delta}_{kj}\{[1+\gamma(n-1)][2+\gamma(n-3)]-f(r,n)-\gamma[1+\gamma(n-2)]\}}{f(r,n)}-\frac{w_{kg}^{E}\gamma[n-1]}{f(r,n)}. \tag{158}$$

(5) implies that:

$$\begin{aligned}
&[1+\gamma(n-1)][2+\gamma(n-3)]-f(r,n)+2[1+\gamma(n-2)][1+\gamma(n-2)]\\
&=[1+\gamma(n-1)][2+\gamma(n-2-1)]-2[1+\gamma(n-2)][2+\gamma(n-2)]\\
&\qquad+\gamma^2[n-1]+2[1+\gamma(n-2)]^2\\
&=[1+\gamma(n-1)][2+\gamma(n-2)-\gamma]-2[1+\gamma(n-2)][2+\gamma(n-2)]\\
&\qquad+\gamma^2[n-1]+2[1+\gamma(n-2)]^2\\
&=[1+\gamma(n-1)][2+\gamma(n-2)]-\gamma[1+\gamma(n-1)]\\
&\qquad-2[1+\gamma(n-2)][2+\gamma(n-2)]+\gamma^2[n-1]+2[1+\gamma(n-2)]^2\\
&=\{1+\gamma(n-1)-2[1+\gamma(n-2)]\}[2+\gamma(n-2)]-\gamma-\gamma^2[n-1]\\
&\qquad+\gamma^2[n-1]+2[1+\gamma(n-2)]^2\\
&=[\gamma(n-1)-1-2\gamma(n-2)][2+\gamma(n-2)]-\gamma+2[1+\gamma(n-2)]^2\\
&=\{\gamma[n-1-2(n-2)]-1\}[2+\gamma(n-2)]-\gamma+2[1+\gamma(n-2)]^2\\
&=[\gamma(3-n)-1][2+\gamma(n-2)]-\gamma+2[1+\gamma(n-2)]^2\\
&=-[1+\gamma(n-3)][2+\gamma(n-2)]-\gamma+2[1+\gamma(n-2)]^2\\
&=-[1+\gamma(n-2-1)][2+\gamma(n-2)]-\gamma+2[1+\gamma(n-2)]^2\\
&=-[1+\gamma(n-2)-\gamma][2+\gamma(n-2)]-\gamma+2[1+\gamma(n-2)]^2
\end{aligned}$$

$$= -[1+\gamma(n-2)][2+\gamma(n-2)] + \gamma[2+\gamma(n-2)] - \gamma + 2[1+\gamma(n-2)]^2$$
$$= \gamma[2+\gamma(n-2)-1] + [1+\gamma(n-2)]\{2[1+\gamma(n-2)] - [2+\gamma(n-2)]\}$$
$$= \gamma[1+\gamma(n-2)] + [1+\gamma(n-2)][2+2\gamma(n-2) - 2 - \gamma(n-2)]$$
$$= \gamma[1+\gamma(n-2)] + \gamma[n-2][1+\gamma(n-2)] = \gamma[n-1][1+\gamma(n-2)]; \qquad (159)$$

$$[1+\gamma(n-1)][2+\gamma(n-3)] - f(r,n) - \gamma[1+\gamma(n-2)]$$
$$= [1+\gamma(n-1)][2+\gamma(n-2-1)] - 2[1+\gamma(n-2)][2+\gamma(n-2)] + \gamma^2[n-1] - \gamma[1+\gamma(n-2)]$$
$$= [1+\gamma(n-1)][2+\gamma(n-2)-\gamma] - 2[1+\gamma(n-2)][2+\gamma(n-2)] + \gamma^2[n-1] - \gamma[1+\gamma(n-2)]$$
$$= [1+\gamma(n-1)][2+\gamma(n-2)] - \gamma[1+\gamma(n-1)] - 2[1+\gamma(n-2)][2+\gamma(n-2)] + \gamma^2[n-1] - \gamma[1+\gamma(n-2)]$$
$$= \{1+\gamma(n-1) - 2[1+\gamma(n-2)]\}[2+\gamma(n-2)] - \gamma - \gamma^2[n-1] + \gamma^2[n-1] - \gamma[1+\gamma(n-2)]$$
$$= [\gamma(n-1) - 1 - 2\gamma(n-2)][2+\gamma(n-2)] - \gamma - \gamma[1+\gamma(n-2)]$$
$$= \{\gamma[n-1-2(n-2)] - 1\}[2+\gamma(n-2)] - \gamma - \gamma[1+\gamma(n-2)]$$
$$= -[1+\gamma(n-3)][2+\gamma(n-2)] - \gamma - \gamma[1+\gamma(n-2)]$$
$$= -[1+\gamma(n-2-1)][2+\gamma(n-2)] - \gamma - \gamma[1+\gamma(n-2)]$$
$$= -[1+\gamma(n-2)-\gamma][2+\gamma(n-2)] - \gamma - \gamma[1+\gamma(n-2)]$$
$$= -[1+\gamma(n-2)][2+\gamma(n-2)] + \gamma[2+\gamma(n-2)] - \gamma - \gamma[1+\gamma(n-2)]$$
$$= -[1+\gamma(n-2)][2+\gamma(n-2)] + \gamma[2+\gamma(n-2) - (1+\gamma(n-2))] - \gamma$$
$$= -[1+\gamma(n-2)][2+\gamma(n-2)] + \gamma - \gamma = -[1+\gamma(n-2)][2+\gamma(n-2)]. \qquad (160)$$

(158), (159), and (160) imply that:

$$q_k^{P-E} = \frac{\Delta_{kj}\gamma[n-1][1+\gamma(n-2)]}{f(r,n)} + \frac{\overline{\Delta}_{kj}[1+\gamma(n-2)][2+\gamma(n-2)]}{f(r,n)} - \frac{w_{kg}^E\gamma[n-1]}{f(r,n)}$$
$$= \frac{[1+\gamma(n-2)]\left\{\Delta_{kj}\gamma[n-1] + \overline{\Delta}_{kj}[2+\gamma(n-2)]\right\} - w_{kg}^E\gamma[n-1]}{f(r,n)}. \blacksquare \qquad (161)$$

Lemma 23. *Suppose Assumption 2 holds. Further suppose Pk ($k \in \{1,2\}$) enters seller market in category g and $n > 1$ downstream sellers in category g ($g \in \{1,2\}$) compete on prices. Then Pk's profit-maximizing commission for each third-party seller in category g is* $\Delta_{kj}\frac{\gamma^4[n-1]^2+4[1+\gamma(n-2)]^3[2+\gamma(n-2)]}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}} + \frac{\gamma\overline{\Delta}_{kj}}{2}$.

Proof. (105) implies that Pk chooses w_{kg}^{E} to:

$$\text{Maximize}\ \ \Pi_k^{P-E} \ = \ \left[\, p_k^{P-E} - c_k^P \,\right] \Theta_k \, q_k^{P-E} \ - F + w_{kg}^E \, \Theta_k \sum_{\substack{j=1 \\ j \neq k}}^{n} q_{kj}^{S-E} \tag{162}$$

$$\Rightarrow \ \ \frac{\partial \Pi_k^{P-E}}{\partial w_{kg}^E} \ = \ \frac{\partial p_k^{P-E}}{\partial w_{kg}^E} \, q_k^{P-E} + \frac{\partial q_k^{P-E}}{\partial w_{kg}^E} \left[\, p_k^{P-E} - c_k^P \,\right] + \sum_{\substack{j=1 \\ j \neq k}}^{n} q_{kj}^{S-E} + w_{kg}^E \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\partial q_{kj}^{S-E}}{\partial w_{kg}^E} \ = \ 0. \tag{163}$$

(132) implies that:

$$\frac{\partial p_k^{P-E}}{\partial w_{kg}^E} \ = \ \frac{\gamma\,[\, n-1\,]\,[\, 3+2\,\gamma\,(\, n-2\,)\,]}{f\,(r,n)}. \tag{164}$$

(143) implies that:

$$\frac{\partial p_{kj}^{S-E}}{\partial w_{kg}^E} \ = \ \frac{\gamma^2\,(\, n-1\,) + 2\,[\, 1+\gamma\,(\, n-2\,)\,]\,[\, 1+\gamma\,(\, n-2\,)\,]}{f\,(r,n)}. \tag{165}$$

(149) implies that:

$$\frac{\partial q_{kj}^{S-E}}{\partial w_{kg}^E} \ = \ - \, \frac{2\,[\, 1+\gamma\,(\, n-2\,)\,]}{f\,(r,n)}. \tag{166}$$

(161) implies that:

$$\frac{\partial q_k^{P-E}}{\partial w_{kg}^E} \ = \ - \, \frac{\gamma\,[\, n-1\,]}{f\,(r,n)}. \tag{167}$$

(132), (149), (161), (163) - (167) imply that:

$$0 = \frac{\gamma\,[\, n-1\,]\,[\, 3+2\,\gamma\,(\, n-2\,)\,]}{f\,(r,n)} \, \frac{[\, 1+\gamma\,(\, n-2\,)\,] \left\{ \Delta_{kj}\gamma\,[n-1] + \overline{\Delta}_{kj}\,[2+\gamma\,(\, n-2\,)\,] \right\} - w_{kg}^E\gamma\,[\, n-1\,]}{f\,(r,n)}$$

$$- \frac{\gamma\,[\, n-1\,]}{f\,(r,n)} \, \frac{[\, 1-\gamma\,]\,[\, 1+\gamma\,(\, n-1\,)\,] \left\{ [2+\gamma\,(\, n-2\,)]\,\overline{\Delta}_{kj} + \gamma\,[\, n-1\,]\,\Delta_{kj} \right\} + \gamma\,[\, n-1\,]\,[3+2\gamma\,(\, n-2\,)\,]\, w_{kg}^E}{f\,(r,n)}$$

$$+ \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{[\, 1+\gamma\,(\, n-2\,)\,] \, \left\{ \Delta_{kj} 2\,[\, 1+\gamma\,(\, n-2\,)\,] + \overline{\Delta}_{kj}\gamma - 2 w_{kg}^E \right\}}{f\,(r,n)} - w_{kg}^E \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{2\,[\, 1+\gamma\,(\, n-2\,)\,]}{f\,(r,n)}$$

$$\Leftrightarrow 0 = \frac{\gamma\,[\, n-1\,]\,[\, 3+2\,\gamma\,(\, n-2\,)\,]}{f\,(r,n)} \, \frac{[\, 1+\gamma\,(\, n-2\,)\,] \left\{ \Delta_{kj}\gamma\,[n-1] + \overline{\Delta}_{kj}\,[2+\gamma\,(\, n-2\,)\,] \right\} - w_{kg}^E\gamma\,[\, n-1\,]}{f\,(r,n)}$$

$$- \frac{\gamma\,[\, n-1\,]}{f\,(r,n)} \, \frac{[\, 1-\gamma\,]\,[\, 1+\gamma\,(\, n-1\,)\,] \left\{ [2+\gamma\,(\, n-2\,)]\,\overline{\Delta}_{kj} + \gamma\,[\, n-1\,]\,\Delta_{kj} \right\} + \gamma\,[\, n-1\,]\,[3+2\gamma\,(\, n-2\,)\,]\, w_{kg}^E}{f\,(r,n)}$$

$$+\frac{[n-1][1+\gamma(n-2)]\ \left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma-2w_{kg}^{E}\right\}}{f(r,n)}-w_{kg}^{E}[n-1]\frac{2[1+\gamma(n-2)]}{f(r,n)} \tag{168}$$

$$\Leftrightarrow w_{kg}^{E}\left\{\frac{\gamma^{2}[n-1]^{2}[3+2\gamma(n-2)]}{[f(r,n)]^{2}}+\frac{\gamma^{2}[n-1]^{2}[3+2\gamma(n-2)]}{[f(r,n)]^{2}}\right.$$
$$\left.+\frac{2[n-1][1+\gamma(n-2)]}{f(r,n)}+\frac{2[n-1][1+\gamma(n-2)]}{f(r,n)}\right\}$$
$$=\frac{\gamma[n-1][3+2\gamma(n-2)]}{f(r,n)}\frac{[1+\gamma(n-2)]\left\{\Delta_{kj}\gamma[n-1]+\overline{\Delta}_{kj}[2+\gamma(n-2)]\right\}}{f(r,n)}$$
$$-\frac{\gamma[n-1]}{f(r,n)}\frac{[1-\gamma][1+\gamma(n-1)]\left\{[2+\gamma(n-2)]\overline{\Delta}_{kj}+\gamma[n-1]\Delta_{kj}\right\}}{f(r,n)}$$
$$+\frac{[n-1][1+\gamma(n-2)]\ \left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma\right\}}{f(r,n)}$$

$$\Leftrightarrow 2w_{kg}^{E}\left\{\gamma^{2}[n-1][3+2\gamma(n-2)]+2f(r,n)[1+\gamma(n-2)]\right\}$$
$$=\gamma[3+2\gamma(n-2)][1+\gamma(n-2)]\left\{\Delta_{kj}\gamma[n-1]+\overline{\Delta}_{kj}[2+\gamma(n-2)]\right\}$$
$$-\gamma[1-\gamma][1+\gamma(n-1)]\left\{[2+\gamma(n-2)]\overline{\Delta}_{kj}+\gamma[n-1]\Delta_{kj}\right\}$$
$$+f(r,n)[1+\gamma(n-2)]\ \left\{\Delta_{kj}2[1+\gamma(n-2)]+\overline{\Delta}_{kj}\gamma\right\}$$

$$\Leftrightarrow 2w_{kg}^{E}\left\{\gamma^{2}[n-1][3+2\gamma(n-2)]+2f(r,n)[1+\gamma(n-2)]\right\}$$
$$=\gamma[3+2\gamma(n-2)][1+\gamma(n-2)]\Delta_{kj}\gamma[n-1]$$
$$+\gamma[3+2\gamma(n-2)][1+\gamma(n-2)]\overline{\Delta}_{kj}[2+\gamma(n-2)]$$
$$-\gamma[1-\gamma][1+\gamma(n-1)][2+\gamma(n-2)]\overline{\Delta}_{kj}$$
$$-\gamma^{2}[1-\gamma][1+\gamma(n-1)][n-1]\Delta_{kj}$$
$$+2[1+\gamma(n-2)]^{2}f(r,n)\ \Delta_{kj}+\gamma f(r,n)[1+\gamma(n-2)]\ \overline{\Delta}_{kj}$$

$$\Leftrightarrow 2w_{kg}^{E}\left\{\gamma^{2}[n-1][3+2\gamma(n-2)]+2f(r,n)[1+\gamma(n-2)]\right\}$$
$$=\Delta_{kj}\left\{\gamma^{2}[n-1][3+2\gamma(n-2)][1+\gamma(n-2)]+2[1+\gamma(n-2)]^{2}f(r,n)\right.$$
$$\left.-\gamma^{2}[1-\gamma][n-1][1+\gamma(n-1)]\right\}$$
$$+\gamma\overline{\Delta}_{kj}\left\{[3+2\gamma(n-2)][1+\gamma(n-2)][2+\gamma(n-2)]+f(r,n)[1+\gamma(n-2)]\right.$$
$$\left.-[1-\gamma][1+\gamma(n-1)][2+\gamma(n-2)]\right\}. \tag{169}$$

(168) reflects $\Delta_{kj} = \Delta_{kh}$ and $\overline{\Delta}_{kj} = \overline{\Delta}_{kh}$ for $\forall j, h \in \{1, 2, ..., n\}$, $j \neq k$, and $h \neq k$ from (116) and (117).

(5) implies that:

$$
\begin{aligned}
&\gamma^2 [n-1] [3+2\gamma(n-2)] [1+\gamma(n-2)] + 2[1+\gamma(n-2)]^2 f(r,n) \\
&\qquad -\gamma^2 [1-\gamma] [n-1] [1+\gamma(n-1)] \\
&= \gamma^2 [n-1] [3+2\gamma(n-2)] [1+\gamma(n-2)] \\
&\qquad +2[1+\gamma(n-2)]^2 \left\{ 2[1+\gamma(n-2)] [2+\gamma(n-2)] - \gamma^2 [n-1] \right\} \\
&\qquad -\gamma^2 [1-\gamma] [n-1] [1+\gamma(n-1)] \\
&= \gamma^2 [n-1] [3+2\gamma(n-2)] [1+\gamma(n-2)] \\
&\qquad +4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] - \gamma^2 [n-1] 2 [1+\gamma(n-2)]^2 \\
&\qquad -\gamma^2 [1-\gamma] [n-1] [1+\gamma(n-1)] \\
&= \gamma^2 [n-1] [1+\gamma(n-2)] \{ 3+2\gamma(n-2) - 2[1+\gamma(n-2)] \} \\
&\qquad +4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] - \gamma^2 [1-\gamma] [n-1] [1+\gamma(n-1)] \\
&= \gamma^2 [n-1] [1+\gamma(n-2)] \{ 3+2\gamma(n-2) - 2 - 2\gamma(n-2) \} \\
&\qquad +4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] - \gamma^2 [1-\gamma] [n-1] [1+\gamma(n-1)] \\
&= \gamma^2 [n-1] [1+\gamma(n-2)] - \gamma^2 [1-\gamma] [n-1] [1+\gamma(n-1)] \\
&\qquad +4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \\
&= \gamma^2 [n-1] \{ 1+\gamma(n-2) - [1-\gamma] [1+\gamma(n-1)] \} \\
&\qquad +4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \\
&= \gamma^2 [n-1] [1+\gamma(n-2) - (1-\gamma) - (1-\gamma)\gamma(n-1)] \\
&\qquad +4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \\
&= \gamma^2 [n-1] [\gamma(n-2) + \gamma - (1-\gamma)\gamma(n-1)] + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \\
&= \gamma^2 [n-1] [\gamma(n-1) - (1-\gamma)\gamma(n-1)] + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \\
&= \gamma^3 [n-1]^2 [1-(1-\gamma)] + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \\
&= \gamma^4 [n-1]^2 + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \; ;
\end{aligned}
\tag{170}
$$

$$
\begin{aligned}
&[3+2\gamma(n-2)] [1+\gamma(n-2)] [2+\gamma(n-2)] + f(r,n) [1+\gamma(n-2)] \\
&\qquad -[1-\gamma] [1+\gamma(n-1)] [2+\gamma(n-2)] \\
&= [3+2\gamma(n-2)] [1+\gamma(n-2)] [2+\gamma(n-2)] \\
&\qquad +[1+\gamma(n-2)] \left\{ 2[1+\gamma(n-2)] [2+\gamma(n-2)] - \gamma^2 [n-1] \right\} \\
&\qquad -[1-\gamma] [1+\gamma(n-1)] [2+\gamma(n-2)] \\
&= [3+2\gamma(n-2)] [1+\gamma(n-2)] [2+\gamma(n-2)]
\end{aligned}
$$

$$
\begin{aligned}
&+2\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&-[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[2+\gamma\,(\,n-2\,)]\\
&=[\,1+\gamma\,(\,n-2\,)\,]\,[2+\gamma\,(\,n-2\,)\,]\,\{\,3+2\,\gamma\,(\,n-2\,)+2\,[\,1+\gamma\,(\,n-2\,)\,]\,\}\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&\quad-[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[2+\gamma\,(\,n-2\,)]\\
&=[\,1+\gamma\,(\,n-2\,)\,]\,[2+\gamma\,(\,n-2\,)\,]\,[\,5+4\,\gamma\,(\,n-2\,)\,]\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&\quad-[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\,[2+\gamma\,(\,n-2\,)]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\{[\,1+\gamma\,(\,n-2\,)\,]\,[\,5+4\,\gamma\,(\,n-2\,)\,]-[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]\}\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left\{5+4\,\gamma^2\,(\,n-2\,)^2+9\,\gamma\,(\,n-2\,)-[\,1+\gamma\,(\,n-1\,)-\gamma\,[\,1+\gamma\,(\,n-1\,)\,]\,]\right\}\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left\{5+4\,\gamma^2\,(\,n-2\,)^2+9\,\gamma\,(\,n-2\,)-\left[\,1+\gamma\,(\,n-1\,)-\gamma-\gamma^2\,(\,n-1\,)\,\right]\right\}\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left[5+4\,\gamma^2\,(\,n-2\,)^2+9\,\gamma\,(\,n-2\,)-1-\gamma\,(\,n-1\,)+\gamma+\gamma^2\,(\,n-1\,)\,\right]\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left[4+4\,\gamma^2\,(\,n-2\,)^2+9\,\gamma\,(\,n-2\,)-\gamma\,(\,n-1-1\,)+\gamma^2\,(\,n-1\,)\,\right]\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left[4+4\,\gamma^2\,(\,n-2\,)^2+9\,\gamma\,(\,n-2\,)-\gamma\,(\,n-2\,)+\gamma^2\,(\,n-1\,)\,\right]\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left[4+4\,\gamma^2\,(\,n-2\,)^2+8\,\gamma\,(\,n-2\,)+\gamma^2\,(\,n-1\,)\,\right]\\
&\quad-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left\{[2+2\gamma\,(\,n-2\,)\,]^2+\gamma^2\,(\,n-1\,)\,\right\}-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=[2+\gamma\,(\,n-2\,)\,]\,\left\{4\,[1+\gamma\,(\,n-2\,)\,]^2+\gamma^2\,(\,n-1\,)\,\right\}-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=4\,[2+\gamma\,(\,n-2\,)\,]\,[1+\gamma\,(\,n-2\,)\,]^2+\gamma^2\,[\,n-1\,]\,[2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=4\,[2+\gamma\,(\,n-2\,)\,]\,[1+\gamma\,(\,n-2\,)\,]^2+\gamma^2\,[\,n-1\,]\,\{2+\gamma\,(\,n-2\,)-[\,1+\gamma\,(\,n-2\,)\,]\,\}\\
&=4\,[2+\gamma\,(\,n-2\,)\,]\,[1+\gamma\,(\,n-2\,)\,]^2+\gamma^2\,[\,n-1\,]\;;
\end{aligned}
\tag{171}
$$

$$
\begin{aligned}
&\gamma^2\,[\,n-1\,]\,[\,3+2\,\gamma\,(\,n-2\,)\,]+2f\,(r,n)\,[\,1+\gamma\,(\,n-2\,)\,]\\
&=\gamma^2\,[\,n-1\,]\,[\,3+2\,\gamma\,(\,n-2\,)\,]+2\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]\right\}[\,1+\gamma\,(\,n-2\,)\,]\\
&=\gamma^2\,[\,n-1\,]\,[\,3+2\,\gamma\,(\,n-2\,)\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]-2\gamma^2\,[\,n-1\,]\,[\,1+\gamma\,(\,n-2\,)\,]
\end{aligned}
$$

$$= \gamma^2 [n-1] \{ 3 + 2\gamma (n-2) - 2 [1 + \gamma (n-2)] \} + 4 [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)]$$
$$= \gamma^2 [n-1] [3 + 2\gamma (n-2) - 2 - 2\gamma (n-2)] + 4 [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)]$$
$$= \gamma^2 [n-1] + 4 [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)] . \tag{172}$$

(169) - (172) imply that:

$$2 w_{kg}^E \Big\{ \gamma^2 [n-1] + 4 [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)] \Big\}$$
$$= \Delta_{kj} \Big\{ \gamma^4 [n-1]^2 + 4 [1 + \gamma (n-2)]^3 [2 + \gamma (n-2)] \Big\}$$
$$+ \gamma \overline{\Delta}_{kj} \Big\{ 4 [2 + \gamma (n-2)] [1 + \gamma (n-2)]^2 + \gamma^2 [n-1] \Big\}$$
$$\Leftrightarrow w_{kg}^E = \Delta_{kj} \frac{\gamma^4 [n-1]^2 + 4 [1 + \gamma (n-2)]^3 [2 + \gamma (n-2)]}{2 \Big\{ \gamma^2 [n-1] + 4 [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)] \Big\}} + \frac{\gamma \overline{\Delta}_{kj}}{2} . \ \blacksquare \tag{173}$$

Lemma 24. *Suppose Assumption 2 holds. Further suppose Pk ($k \in \{1, 2\}$) enters seller market in category g and $n > 1$ downstream sellers in category g ($g \in \{1, 2\}$) compete on prices. Then each seller's (Sj) equilibrium output (Q_{kj}^{S-E}) is* $\frac{\Theta_k \Delta_{kj} [1+\gamma(n-2)] \{ 2[1+\gamma(n-2)]^2 + \gamma^2 [n-1] \}}{\gamma^2 [n-1] + 4 [1+\gamma(n-2)]^2 [2+\gamma(n-2)]}$, *Sj's profit is* $\Theta_k [1-\gamma] [1 + \gamma (n-1)] [1 + \gamma (n-2)] \left\{ \frac{\Delta_{kj} \{ 2[1+\gamma(n-2)]^2 + \gamma^2 [n-1] \}}{\gamma^2 [n-1] + 4 [1+\gamma(n-2)]^2 [2+\gamma(n-2)]} \right\}^2$, *Pk's equilibrium output (q_k^{P-E}) is* $\frac{1}{2} \left\{ \frac{\Delta_{kj} \gamma [n-1] \{ 2[1+\gamma(n-2)]^2 + \gamma^2 [n-1] \}}{\gamma^2 [n-1] + 4 [1+\gamma(n-2)]^2 [2+\gamma(n-2)]} + \overline{\Delta}_{kj} \right\}$, *and Pk's profit from the commission it collects from sellers and from entering seller market is* $\Theta_k H_{kj} - F$.

<u>Proof.</u> (5), (149) and Lemma 23 imply that:

$$q_{kj}^{S-E} = \frac{[1 + \gamma (n-2)]}{f(r, n)}$$
$$\cdot \Big\{ \Delta_{kj} 2 [1 + \gamma (n-2)] + \overline{\Delta}_{kj} \gamma$$
$$- \Delta_{kj} \frac{\gamma^4 [n-1]^2 + 4 [1 + \gamma (n-2)]^3 [2 + \gamma (n-2)]}{\gamma^2 [n-1] + 4 [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)]} - \gamma \overline{\Delta}_{kj} \Big\}$$
$$= \frac{[1 + \gamma (n-2)]}{f(r, n)}$$
$$\cdot \Big\{ \Delta_{kj} 2 [1 + \gamma (n-2)] - \Delta_{kj} \frac{\gamma^4 [n-1]^2 + 4 [1 + \gamma (n-2)]^3 [2 + \gamma (n-2)]}{\gamma^2 [n-1] + 4 [1 + \gamma (n-2)]^2 [2 + \gamma (n-2)]} \Big\}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,]\,\Delta_{kj}}{f\,(r,n)\left\{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]\,\{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\}\right.$$

$$\left.-\gamma^4\,[n-1]^2-4\,[\,1+\gamma\,(\,n-2\,)\,]^3\,[\,2+\gamma\,(\,n-2\,)\,]\right\}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,]\,\Delta_{kj}}{f\,(r,n)\left\{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]\,\gamma^2\,[\,n-1\,]+8\,[\,1+\gamma\,(\,n-2\,)\,]^3\,[\,2+\gamma\,(\,n-2\,)\,]\right.$$

$$\left.-\gamma^4\,[n-1]^2-4\,[\,1+\gamma\,(\,n-2\,)\,]^3\,[\,2+\gamma\,(\,n-2\,)\,]\right\}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,]\,\Delta_{kj}}{f\,(r,n)\left\{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]\,\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^3\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^4\,[n-1]^2\right\}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,]\,\Delta_{kj}}{f\,(r,n)\left\{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]\,\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^3\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^4\,[n-1]^2\right\}$$

$$= \frac{[\,1+\gamma\,(\,n-2\,)\,]\,\Delta_{kj}}{f\,(r,n)\left\{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,]-\gamma^2\,[\,n-1\,]\right\}\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]^2+\gamma^2\,[\,n-1\,]\right\}$$

$$= \frac{\Delta_{kj}\,[\,1+\gamma\,(\,n-2\,)\,]\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]^2+\gamma^2\,[\,n-1\,]\right\}}{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]}. \tag{174}$$

(174) reflects (5).

(150) and (174) imply that:

$$\pi_{kj}^{S-E} = \frac{\Theta_k\,[\,1-\gamma\,]\,[\,1+\gamma\,(\,n-1\,)\,]}{[\,1+\gamma\,(\,n-2\,)\,]}\left\{\frac{\Delta_{kj}\,[\,1+\gamma\,(\,n-2\,)\,]\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]^2+\gamma^2\,[\,n-1\,]\right\}}{\gamma^2\,[\,n-1\,]+4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]}\right\}^2$$

$$= \Theta_k [1-\gamma][1+\gamma(n-1)][1+\gamma(n-2)] \left\{ \frac{\Delta_{kj} \left\{2[1+\gamma(n-2)]^2 + \gamma^2 [n-1]\right\}}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]} \right\}^2 . \tag{175}$$

(5), (161) and Lemma 23 imply that:

$$q_k^{P-E} = \frac{1}{f(r,n)} \left\{ [1+\gamma(n-2)] \left\{\Delta_{kj} \gamma [n-1] + \overline{\Delta}_{kj} [2+\gamma(n-2)]\right\} \right.$$
$$\left. - \frac{\gamma[n-1]}{2} \left[\Delta_{kj} \frac{\gamma^4 [n-1]^2 + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)]}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]} + \gamma \overline{\Delta}_{kj} \right] \right\}$$
$$= \frac{1}{f(r,n)} \left\{ \Delta_{kj} \gamma [n-1][1+\gamma(n-2)] + \overline{\Delta}_{kj} [1+\gamma(n-2)][2+\gamma(n-2)] \right.$$
$$\left. - \frac{\Delta_{kj} \gamma [n-1]}{2} \frac{\gamma^4 [n-1]^2 + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)]}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]} - \frac{\gamma^2 [n-1]}{2} \overline{\Delta}_{kj} \right\}$$
$$= \frac{1}{2f(r,n)} \left\{ \Delta_{kj} \gamma [n-1] \left[2[1+\gamma(n-2)] - \frac{\gamma^4 [n-1]^2 + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)]}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]} \right] \right.$$
$$\left. + \overline{\Delta}_{kj} \left\{2[1+\gamma(n-2)][2+\gamma(n-2)] - \gamma^2 [n-1]\right\} \right\}$$
$$= \frac{1}{2f(r,n)} \left\{ \Delta_{kj} \gamma [n-1] \left[2[1+\gamma(n-2)] - \frac{\gamma^4 [n-1]^2 + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)]}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]} \right] \right.$$
$$\left. + \overline{\Delta}_{kj} f(r,n) \right\}. \tag{176}$$

Observe that:

$$2[1+\gamma(n-2)] - \frac{\gamma^4 [n-1]^2 + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)]}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]}$$
$$= \frac{1}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]}$$
$$\cdot \left\{ 2[1+\gamma(n-2)] \gamma^2 [n-1] + 8[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \right.$$
$$\left. - \gamma^4 [n-1]^2 - 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] \right\}$$
$$= \frac{2[1+\gamma(n-2)] \gamma^2 [n-1] + 4[1+\gamma(n-2)]^3 [2+\gamma(n-2)] - \gamma^4 [n-1]^2}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]}$$
$$= \frac{\left\{2[1+\gamma(n-2)][2+\gamma(n-2)] - \gamma^2 [n-1]\right\} \left\{2[1+\gamma(n-2)]^2 + \gamma^2 [n-1]\right\}}{\gamma^2 [n-1] + 4[1+\gamma(n-2)]^2 [2+\gamma(n-2)]}$$

$$= \frac{f(r,n)\left\{2\left[1+\gamma(n-2)\right]^2+\gamma^2\left[n-1\right]\right\}}{\gamma^2\left[n-1\right]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]}. \tag{177}$$

(176) and (177) reflect (5).

(176) and (177) imply that:

$$q_k^{P-E} = \frac{1}{2f(r,n)}\left\{\Delta_{kj}\gamma\left[n-1\right]\frac{f(r,n)\left\{2\left[1+\gamma(n-2)\right]^2+\gamma^2\left[n-1\right]\right\}}{\gamma^2\left[n-1\right]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]}+\overline{\Delta}_{kj}f(r,n)\right\}$$

$$= \frac{1}{2}\left\{\frac{\Delta_{kj}\gamma\left[n-1\right]\left\{2\left[1+\gamma(n-2)\right]^2+\gamma^2\left[n-1\right]\right\}}{\gamma^2\left[n-1\right]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]}+\overline{\Delta}_{kj}\right\}. \tag{178}$$

(5), (132) and Lemma 23 imply that:

$$p_k^{P-E}-c_k^P$$

$$= \frac{1}{f(r,n)}\Bigg\{\left[1-\gamma\right]\left[1+\gamma(n-1)\right]\left\{\left[2+\gamma(n-2)\right]\overline{\Delta}_{kj}+\gamma\left[n-1\right]\Delta_{kj}\right\}$$
$$+\frac{\gamma\left[n-1\right]\left[3+2\gamma(n-2)\right]}{2}$$
$$\cdot\left[\Delta_{kj}\frac{\gamma^4\left[n-1\right]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]}{\gamma^2\left[n-1\right]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]}+\gamma\overline{\Delta}_{kj}\right]\Bigg\}$$

$$= \frac{1}{f(r,n)}\Bigg\{\left[1-\gamma\right]\left[1+\gamma(n-1)\right]\left[2+\gamma(n-2)\right]\overline{\Delta}_{kj}+\gamma\left[n-1\right]\left[1-\gamma\right]\left[1+\gamma(n-1)\right]\Delta_{kj}$$
$$+\frac{\gamma\left[n-1\right]\left[3+2\gamma(n-2)\right]}{2}\Delta_{kj}\frac{\gamma^4\left[n-1\right]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]}{\gamma^2\left[n-1\right]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]}$$
$$+\frac{\gamma\left[n-1\right]\left[3+2\gamma(n-2)\right]}{2}\gamma\overline{\Delta}_{kj}\Bigg\}$$

$$= \frac{1}{f(r,n)}\Bigg\{\overline{\Delta}_{kj}\left[\left[1-\gamma\right]\left[1+\gamma(n-1)\right]\left[2+\gamma(n-2)\right]+\frac{\gamma^2\left[n-1\right]\left[3+2\gamma(n-2)\right]}{2}\right]$$
$$+\gamma\left[n-1\right]\Delta_{kj}$$
$$\cdot\Bigg[(1-\gamma)(1+\gamma\left[n-1\right])$$
$$+\frac{\left[3+2\gamma(n-2)\right]}{2}\frac{\gamma^4\left[n-1\right]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]}{\gamma^2\left[n-1\right]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]}\Bigg]\Bigg\}. \tag{179}$$

Observe that:

$$\left[1-\gamma\right]\left[1+\gamma(n-1)\right]\left[2+\gamma(n-2)\right]+\frac{\gamma^2\left[n-1\right]\left[3+2\gamma(n-2)\right]}{2}$$

$$= [1-\gamma][1+\gamma(n-1)][2+\gamma(n-2)] + \frac{\gamma^2[n-1][1+\gamma(n-2)+2+\gamma(n-2)]}{2}$$

$$= [1-\gamma][1+\gamma(n-1)][2+\gamma(n-2)] + \frac{\gamma^2[n-1][2+\gamma(n-2)]}{2} + \frac{\gamma^2[n-1][1+\gamma(n-2)]}{2}$$

$$= [2+\gamma(n-2)]\left[[1-\gamma][1+\gamma(n-1)] + \frac{\gamma^2(n-1)}{2}\right] + \frac{\gamma^2[n-1][1+\gamma(n-2)]}{2}$$

$$= [2+\gamma(n-2)]\left[1+\gamma(n-1)-\gamma-\gamma^2(n-1) + \frac{\gamma^2(n-1)}{2}\right] + \frac{\gamma^2[n-1][1+\gamma(n-2)]}{2}$$

$$= [2+\gamma(n-2)]\left[1+\gamma(n-1-1) - \frac{\gamma^2(n-1)}{2}\right] + \frac{\gamma^2[n-1][1+\gamma(n-2)]}{2}$$

$$= [2+\gamma(n-2)]\left[1+\gamma(n-2) - \frac{\gamma^2(n-1)}{2}\right] + \frac{\gamma^2[n-1][1+\gamma(n-2)]}{2}$$

$$= [2+\gamma(n-2)][1+\gamma(n-2)] - \frac{\gamma^2[n-1][2+\gamma(n-2)]}{2} + \frac{\gamma^2[n-1][1+\gamma(n-2)]}{2}$$

$$= [2+\gamma(n-2)][1+\gamma(n-2)] + \frac{\gamma^2[n-1]\{1+\gamma(n-2)-[2+\gamma(n-2)]\}}{2}$$

$$= [2+\gamma(n-2)][1+\gamma(n-2)] - \frac{\gamma^2[n-1]}{2} = \frac{f(r,n)}{2};$$

(180)

$$[1-\gamma][1+\gamma(n-1)] + \frac{[3+2\gamma(n-2)]}{2}\frac{\gamma^4[n-1]^2+4[1+\gamma(n-2)]^3[2+\gamma(n-2)]}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]}$$

$$= 1+\gamma[n-1]-\gamma-\gamma^2[n-1] + \frac{[3+2\gamma(n-2)]}{2}\frac{\gamma^4[n-1]^2+4[1+\gamma(n-2)]^3[2+\gamma(n-2)]}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]}$$

$$= 1+\gamma[n-1-1]-\gamma^2[n-1] + \frac{[3+2\gamma(n-2)]}{2}\frac{\gamma^4[n-1]^2+4[1+\gamma(n-2)]^3[2+\gamma(n-2)]}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]}$$

$$= 1+\gamma[n-2]-\gamma^2[n-1] + \frac{[3+2\gamma(n-2)]}{2}\frac{\gamma^4[n-1]^2+4[1+\gamma(n-2)]^3[2+\gamma(n-2)]}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]}$$

$$= \frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$

$$\cdot\Big\{2\left[1+\gamma(n-2)-\gamma^2(n-1)\right]\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}$$

$$+[3+2\gamma(n-2)]\left\{\gamma^4[n-1]^2+4[1+\gamma(n-2)]^3[2+\gamma(n-2)]\right\}\Big\}$$

$$= \frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$

$$\cdot\Big\{2[1+\gamma(n-2)]\gamma^2[n-1]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]$$

$$-2\gamma^4[n-1]^2 - 8\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$+[3+2\gamma(n-2)]\gamma^4[n-1]^2$$
$$\left.+4[3+2\gamma(n-2)][1+\gamma(n-2)]^3[2+\gamma(n-2)]\right\}$$

$$= \frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\left\{2[1+\gamma(n-2)]\gamma^2[n-1]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]\right.$$
$$-2\gamma^4[n-1]^2 - 8\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$+[1+\gamma(n-2)+2+\gamma(n-2)]\gamma^4[n-1]^2$$
$$\left.+4[1+\gamma(n-2)+2+\gamma(n-2)][1+\gamma(n-2)]^3[2+\gamma(n-2)]\right\}$$

$$= \frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\left\{2\gamma^2[n-1][1+\gamma(n-2)]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]\right.$$
$$-2\gamma^4[n-1]^2 - 8\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$+[1+\gamma(n-2)]\gamma^4[n-1]^2+[2+\gamma(n-2)]\gamma^4[n-1]^2$$
$$\left.+4[1+\gamma(n-2)]^4[2+\gamma(n-2)]+4[1+\gamma(n-2)]^3[2+\gamma(n-2)]^2\right\}$$

$$= \frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\left\{2\gamma^2[n-1][1+\gamma(n-2)]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]\right.$$
$$-2\gamma^4[n-1]^2 - 6\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$+[1+\gamma(n-2)]\gamma^4[n-1]^2+[2+\gamma(n-2)]\gamma^4[n-1]^2$$
$$+4[1+\gamma(n-2)]^4[2+\gamma(n-2)]$$
$$+2[1+\gamma(n-2)]^2[2+\gamma(n-2)]\left\{2[1+\gamma(n-2)][2+\gamma(n-2)]-\gamma^2[n-1]\right\}$$

$$= \frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\left\{2\gamma^2[n-1][1+\gamma(n-2)]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]\right.$$
$$-2\gamma^4[n-1]^2 - 4\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$

$$+[1+\gamma(n-2)]\gamma^4[n-1]^2-2\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]+[2+\gamma(n-2)]\gamma^4[n-1]^2$$
$$+4[1+\gamma(n-2)]^4[2+\gamma(n-2)]+2[1+\gamma(n-2)]^2[2+\gamma(n-2)]f(r,n)\Big\} \quad (181)$$

$$=\frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\Big\{2\gamma^2[n-1][1+\gamma(n-2)]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]$$
$$-2\gamma^4[n-1]^2-4\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$+[1+\gamma(n-2)]\gamma^2[n-1]\left\{\gamma^2[n-1]-2[1+\gamma(n-2)][2+\gamma(n-2)]\right\}+[2+\gamma(n-2)]\gamma^4[n-1]^2$$
$$+4[1+\gamma(n-2)]^4[2+\gamma(n-2)]+2[1+\gamma(n-2)]^2[2+\gamma(n-2)]f(r,n)\Big\}$$

$$=\frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\Big\{2\gamma^2[n-1][1+\gamma(n-2)]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]$$
$$-2\gamma^4[n-1]^2-4\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$-[1+\gamma(n-2)]\gamma^2[n-1]f(r,n)+[2+\gamma(n-2)]\gamma^4[n-1]^2$$
$$+4[1+\gamma(n-2)]^4[2+\gamma(n-2)]+2[1+\gamma(n-2)]^2[2+\gamma(n-2)]f(r,n)\Big\} \quad (182)$$

$$=\frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\Big\{2\gamma^2[n-1][1+\gamma(n-2)]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]$$
$$-\gamma^4[n-1]^2-4\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$-[1+\gamma(n-2)]\gamma^2[n-1]f(r,n)+[2+\gamma(n-2)-1]\gamma^4[n-1]^2$$
$$+4[1+\gamma(n-2)]^4[2+\gamma(n-2)]+2[1+\gamma(n-2)]^2[2+\gamma(n-2)]f(r,n)\Big\}$$

$$=\frac{1}{2\left\{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]\right\}}$$
$$\cdot\Big\{2\gamma^2[n-1][1+\gamma(n-2)]+8[1+\gamma(n-2)]^3[2+\gamma(n-2)]$$
$$-\gamma^4[n-1]^2-4\gamma^2[n-1][1+\gamma(n-2)]^2[2+\gamma(n-2)]$$
$$-[1+\gamma(n-2)]\gamma^2[n-1]f(r,n)+[1+\gamma(n-2)]\gamma^4[n-1]^2$$

$$+4\left[1+\gamma\left(n-2\right)\right]^{4}\left[2+\gamma\left(n-2\right)\right]+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\}$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$-\gamma^{4}\left[n-1\right]^{2}-2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+\left[1+\gamma\left(n-2\right)\right]\gamma^{4}\left[n-1\right]^{2}-2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]$$

$$+4\left[1+\gamma\left(n-2\right)\right]^{4}\left[2+\gamma\left(n-2\right)\right]+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\}$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$-\gamma^{4}\left[n-1\right]^{2}-2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]\left\{\gamma^{2}\left[n-1\right]-2\left[1+\gamma\left(n-2\right)\right]\left[2+\gamma\left(n-2\right)\right]\right\}$$

$$+4\left[1+\gamma\left(n-2\right)\right]^{4}\left[2+\gamma\left(n-2\right)\right]+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\}$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$-\gamma^{4}\left[n-1\right]^{2}-2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+4\left[1+\gamma\left(n-2\right)\right]^{4}\left[2+\gamma\left(n-2\right)\right]+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\} \quad (183)$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]-\gamma^{4}\left[n-1\right]^{2}+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\left\{2\left[1+\gamma\left(n-2\right)\right]^{2}-\gamma^{2}\left[n-1\right]\right\}$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\}$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]-\gamma^{4}\left[n-1\right]^{2}+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\left\{2\left[1+\gamma\left(n-2\right)\right]\left[2-1+\gamma\left(n-2\right)\right]-\gamma^{2}\left[n-1\right]\right\}$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\}$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]-\gamma^{4}\left[n-1\right]^{2}+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\left\{2\left[1+\gamma\left(n-2\right)\right]\left[2+\gamma\left(n-2\right)\right]-\gamma^{2}\left[n-1\right]\right\}$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]2\left[1+\gamma\left(n-2\right)\right]\left[-1\right]$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\}$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]-\gamma^{4}\left[n-1\right]^{2}+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)$$

$$-4\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+2\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]f\left(r,n\right)\Bigg\} \tag{184}$$

$$=\frac{1}{2\left\{\gamma^{2}\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^{2}\left[2+\gamma\left(n-2\right)\right]\right\}}$$

$$\cdot\Bigg\{2\gamma^{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]-\gamma^{4}\left[n-1\right]^{2}+8\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$-4\left[1+\gamma\left(n-2\right)\right]^{3}\left[2+\gamma\left(n-2\right)\right]$$

$$-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)-\left[1+\gamma\left(n-2\right)\right]\gamma^{2}\left[n-1\right]f\left(r,n\right)$$

$$+2\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]f(r,n)+2\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]f(r,n)\Big\}$$

$$=\frac{1}{2\left\{\gamma^2[n-1]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]\right\}}$$

$$\cdot\Big\{2\gamma^2[n-1]\left[1+\gamma(n-2)\right]-\gamma^4[n-1]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]$$

$$-2\left[1+\gamma(n-2)\right]\gamma^2[n-1]f(r,n)+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]f(r,n)\Big\}$$

$$=\frac{1}{2\left\{\gamma^2[n-1]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]\right\}}$$

$$\cdot\Big\{2\gamma^2[n-1]\left[1+\gamma(n-2)\right]-\gamma^4[n-1]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]$$

$$-2\left[1+\gamma(n-2)\right]\gamma^2[n-1]f(r,n)+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]f(r,n)\Big\}. \quad (185)$$

Observe that:

$$2\gamma^2[n-1]\left[1+\gamma(n-2)\right]-\gamma^4[n-1]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]$$
$$=2\gamma^2[n-1]\left[1+\gamma(n-2)\right]-\gamma^2[n-1]\,2\left[1+\gamma(n-2)\right]\left[2+\gamma(n-2)\right]$$
$$+\gamma^2[n-1]\,2\left[1+\gamma(n-2)\right]\left[2+\gamma(n-2)\right]-\gamma^4[n-1]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]$$
$$=2\gamma^2[n-1]\left[1+\gamma(n-2)\right]-\gamma^2[n-1]\,2\left[1+\gamma(n-2)\right]\left[2+\gamma(n-2)\right]$$
$$+\gamma^2[n-1]\left\{2\left[1+\gamma(n-2)\right]\left[2+\gamma(n-2)\right]-\gamma^2[n-1]\right\}+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]$$
$$=2\gamma^2[n-1]\left[1+\gamma(n-2)\right]\left\{1-\left[2+\gamma(n-2)\right]\right\}$$
$$+\gamma^2[n-1]f(r,n)+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right] \quad (186)$$
$$=-2\gamma^2[n-1]\left[1+\gamma(n-2)\right]\left[1+\gamma(n-2)\right]$$
$$+\gamma^2[n-1]f(r,n)+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]$$
$$=-2\gamma^2[n-1]\left[1+\gamma(n-2)\right]^2$$
$$+\gamma^2[n-1]f(r,n)+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]$$
$$=2\left[1+\gamma(n-2)\right]^2\left\{2\left[1+\gamma(n-2)\right]\left[2+\gamma(n-2)\right]-\gamma^2[n-1]\right\}+\gamma^2[n-1]f(r,n)$$
$$=2\left[1+\gamma(n-2)\right]^2f(r,n)+\gamma^2[n-1]f(r,n). \quad (187)$$

(185) and (187) imply that:

$$[1-\gamma]\left[1+\gamma(n-1)\right]+\frac{\left[3+2\gamma(n-2)\right]}{2}\frac{\gamma^4[n-1]^2+4\left[1+\gamma(n-2)\right]^3\left[2+\gamma(n-2)\right]}{\gamma^2[n-1]+4\left[1+\gamma(n-2)\right]^2\left[2+\gamma(n-2)\right]}$$

$$= \frac{1}{2\left\{\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\Bigg\{2\,[\,1+\gamma\,(\,n-2\,)]^2\,f\,(r,n) + \gamma^2\,[\,n-1\,]\,f\,(r,n)$$

$$-2\,[1+\gamma\,(\,n-2\,)\,]\,\gamma^2\,[\,n-1\,]\,f\,(r,n) + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\,f\,(r,n)\Bigg\}$$

$$= \frac{f\,(r,n)}{2\left\{\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\Bigg\{2\,[\,1+\gamma\,(\,n-2\,)]^2 + \gamma^2\,[\,n-1\,]$$

$$-2\,[1+\gamma\,(\,n-2\,)\,]\,\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\Bigg\}$$

$$= \frac{f\,(r,n)}{2\left\{\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}$$

$$\cdot\Bigg\{2\,[\,1+\gamma\,(\,n-2\,)]^2 + \gamma^2\,[\,n-1\,]$$

$$+2\,[1+\gamma\,(\,n-2\,)\,]\left\{2\,[\,1+\gamma\,(\,n-2\,)\,]\,[\,2+\gamma\,(\,n-2\,)\,] - \gamma^2\,[\,n-1\,]\right\}\Bigg\}$$

$$= \frac{f\,(r,n)\Bigg\{2\,[\,1+\gamma\,(\,n-2\,)]^2 + \gamma^2\,[\,n-1\,] + 2\,[1+\gamma\,(\,n-2\,)\,]\,f\,(r,n)\Bigg\}}{2\left\{\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}. \tag{188}$$

(180) - (188) reflect (5).

(179), (180), and (188) imply that:

$$p_k^{P-E} - c_k^P$$

$$= \frac{1}{f\,(r,n)}\Bigg\{\overline{\Delta}_{kj}\frac{f\,(r,n)}{2}$$

$$+\gamma\,[\,n-1\,]\,\Delta_{kj}\frac{f\,(r,n)\left\{2\,[\,1+\gamma\,(\,n-2\,)]^2 + \gamma^2\,[\,n-1\,] + 2\,[1+\gamma\,(\,n-2\,)\,]\,f\,(r,n)\right\}}{2\left\{\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}\Bigg\}$$

$$= \frac{\overline{\Delta}_{kj}}{2} + \frac{\Delta_{kj}\gamma\,[\,n-1\,]\left\{2\,[\,1+\gamma\,(\,n-2\,)]^2 + \gamma^2\,[\,n-1\,] + 2\,[1+\gamma\,(\,n-2\,)\,]\,f\,(r,n)\right\}}{2\left\{\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}. \tag{189}$$

(162), (174), (178), (189), and Lemma 23 imply that:

$$\Pi_k^{P-E} = \frac{\Theta_k}{2}\left\{\frac{\overline{\Delta}_{kj}}{2} + \frac{\Delta_{kj}\gamma\,[\,n-1\,]\left\{2\,[\,1+\gamma\,(\,n-2\,)]^2 + \gamma^2\,[\,n-1\,] + 2\,[1+\gamma\,(\,n-2\,)\,]\,f\,(r,n)\right\}}{2\left\{\gamma^2\,[\,n-1\,] + 4\,[\,1+\gamma\,(\,n-2\,)\,]^2\,[\,2+\gamma\,(\,n-2\,)\,]\right\}}\right\}$$

$$\cdot\left\{\frac{\Delta_{kj}\gamma\left[n-1\right]\left\{2\left[1+\gamma\left(n-2\right)\right]^2+\gamma^2\left[n-1\right]\right\}}{\gamma^2\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^2\left[2+\gamma\left(n-2\right)\right]}+\overline{\Delta}_{kj}\right\}$$

$$+\Theta_k\left\{\Delta_{kj}\frac{\gamma^4\left[n-1\right]^2+4\left[1+\gamma\left(n-2\right)\right]^3\left[2+\gamma\left(n-2\right)\right]}{2\left\{\gamma^2\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^2\left[2+\gamma\left(n-2\right)\right]\right\}}+\frac{\gamma\overline{\Delta}_{kj}}{2}\right\}$$

$$\cdot\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{\Delta_{kj}\left[1+\gamma\left(n-2\right)\right]\left\{2\left[1+\gamma\left(n-2\right)\right]^2+\gamma^2\left[n-1\right]\right\}}{\gamma^2\left[n-1\right]+4\left[1+\gamma\left(n-2\right)\right]^2\left[2+\gamma\left(n-2\right)\right]}-F$$

$$=\frac{\Theta_k}{2}\left\{\frac{\overline{\Delta}_{kj}}{2}+\frac{\Delta_{kj}\gamma\left[n-1\right]\left\{2\left[1+\gamma\left(n-2\right)\right]^2+\gamma^2\left[n-1\right]+2\left[1+\gamma\left(n-2\right)\right]f\left(r,n\right)\right\}}{2g\left(r,n\right)}\right\}$$

$$\cdot\left\{\frac{\Delta_{kj}\gamma\left[n-1\right]\left\{2\left[1+\gamma\left(n-2\right)\right]^2+\gamma^2\left[n-1\right]\right\}}{g\left(r,n\right)}+\overline{\Delta}_{kj}\right\}$$

$$+\Theta_k\left\{\Delta_{kj}\frac{\gamma^4\left[n-1\right]^2+4\left[1+\gamma\left(n-2\right)\right]^3\left[2+\gamma\left(n-2\right)\right]}{2g\left(r,n\right)}+\frac{\gamma\overline{\Delta}_{kj}}{2}\right\}$$

$$\cdot\left[n-1\right]\frac{\Delta_{kj}\left[1+\gamma\left(n-2\right)\right]\left\{2\left[1+\gamma\left(n-2\right)\right]^2+\gamma^2\left[n-1\right]\right\}}{g\left(r,n\right)}-F \tag{190}$$

$$=\Theta_k H_{kj}-F, \tag{191}$$

where H_{kj} is given by (7), and (190) reflects (6) and $\Delta_{kj}=\Delta_{kh}$ for $\forall j,h\in\{1,2,...,n\}$, $j\neq k$, and $h\neq k$ from (116). ■

Condition E $F\in(F_1,\min\{F_2,F_3\})$, where $F_1\equiv\Theta_2H_{22}-\frac{\Theta_2 n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{21}\right)^2+\left(\widetilde{\Delta}_{22}\right)^2\right]}{4[2+\gamma(n-3)]}$, $F_2\equiv\Theta_kH_{kj}-\frac{\Theta_k n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kj}\right]^2}{4[2+\gamma(n-3)]}$, and $F_3\equiv\Theta_1H_{11}-\frac{\Theta_1 n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{11}\right)^2+\left(\widetilde{\Delta}_{12}\right)^2\right]}{4[2+\gamma(n-3)]}$.

Condition E ensures that platform entry is feasible in this setting.

Monopoly Platform (MP).

Proposition 5. *Suppose Condition E holds. In the monopolistic platform setting, both sellers sell on the platform (e.g., Pk) and the platform enters both sellers' product markets in equilibrium. Sj's equilibrium profit is*

$\Theta_k\left[1-\gamma\right]\left[1+\gamma\left(n-1\right)\right]\left[1+\gamma\left(n-2\right)\right]\left\{\frac{\Delta_{kj}\left\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\right\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]}\right\}^2$, and the platform's equilibrium profit is $\Theta_kH_{kj}-F$.

Proof. Lemma 21 implies that Pk's profit is $\frac{\Theta_k n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kj}\right]^2}{4[2+\gamma(n-3)]}$, if S$j$ ($j\in\{1,2,...,n\}$) in category g ($g\in\{1,2\}$) sells product j on Pk and Pk does not enter Sj's product market. Lemma 24 implies that Pk's profit is $\Theta_kH_{kj}-F$, if Sj sells product j on Pk and Pk enters Sj's product market. Because Condition E holds, $\Theta_kH_{kj}-F>\frac{\Theta_k n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kj}\right]^2}{4[2+\gamma(n-3)]}$, i.e., P$k$ secures a higher profit by entering Sj's market than "no entry". Therefore, if Sj sells on

Pk, Pk will enter Sj's market, Lemma 24 implies that Pk's profit is $\Theta_k H_{kj} - F$ and Sj's profit is $\Theta_k [1-\gamma][1+\gamma(n-1)][1+\gamma(n-2)] \left\{ \frac{\Delta_{kj}\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]} \right\}^2$. Therefore, knowing P$k$'s entry decisions, S$j$ will choose to sell on Pk because he secures a positive profit if he sells on Pk (i.e., $\Theta_k [1-\gamma][1+\gamma(n-1)][1+\gamma(n-2)] \left\{ \frac{\Delta_{kj}\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]} \right\}^2 > 0$) while he secures zero profit if he does not sell on Pk, regardless of the other seller's choice. Therefore, in equilibrium, both S1 and S2 sell on Pk, and Pk enters each seller's market. ■

Platform Competition (PC).

Lemma 25. *Suppose Assumption 2 holds. Further suppose both platforms commit not to enter. Then sellers in category g will: (i) sell on Pi if $\Theta_i > \Theta_k$ $(g, i, k \in \{1, 2\})$; and (ii) sell on Pk if $\Theta_i < \Theta_k$.*

Proof. Lemma 21 implies that Sj's profit is $\frac{\Theta_k[1-\gamma][1+\gamma(n-2)][1+\gamma(n-1)][\widetilde{\Delta}_{kj}]^2}{4[2+\gamma(n-3)]^2}$ if Sj sells on Pk ($j \in \{1, 2, ..., n\}, k \in \{1, 2\}$). Assumption 2 and (86) imply that $\widetilde{\Delta}_{kj} = \widetilde{\Delta}_{ij}$ for $\forall j \in \{1, 2, ..., n\}$. Consequently, if $\Theta_1 = \Theta_2$, then Sj is indifferent between selling on P1 and selling on P2; if $\Theta_i > \Theta_k$, then Sj sells on Pi. ■

Lemma 26. *Suppose Assumption 2 and Condition E hold. Further suppose platforms both make no commitment. Then sellers in category g will: (i) sell on P1 when $\frac{\Theta_1}{\Theta_2} > \left[\frac{\Delta_{2g}}{\Delta_{1g}}\right]^2$; and (ii) sell on P2 when $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{2g}}{\Delta_{1g}}\right]^2$ $(g, i, k \in \{1, 2\})$.*

Proof. Condition E implies that Pk will enter seller market if Sj sells on Pk. Lemma 24 implies that Sj's profit is $\Theta_k [1-\gamma][1+\gamma(n-1)][1+\gamma(n-2)] \left\{ \frac{\Delta_{kjg}\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]} \right\}^2$ if Sj sells on Pk ($j \in \{1, 2, ..., n\}, k \in \{1, 2\}$). Therefore, for $j \in \{1, 2, ..., n\}, i, k \in \{1, 2\}$, S$j$ in category g will sell on Pi if

$$\Theta_i [1-\gamma][1+\gamma(n-1)][1+\gamma(n-2)] \left\{ \frac{\Delta_{ijg}\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]} \right\}^2$$

$$> \Theta_k [1-\gamma][1+\gamma(n-1)][1+\gamma(n-2)] \left\{ \frac{\Delta_{kjg}\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]} \right\}^2$$

$$\Leftrightarrow \frac{\Theta_i}{\Theta_k} > \left[\frac{\Delta_{kg}}{\Delta_{ig}}\right]^2. \blacksquare$$

It can be shown that for $\gamma \in (0, 1)$ and $n > 2$:

$$\xi(\gamma, n) \equiv \frac{2[2+\gamma(n-3)]\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]} < 1. \tag{192}$$

Lemma 27. *Suppose Assumption 2 holds. Further suppose Pk commits not to enter and Pi makes no commitment* ($i, k \in \{1, 2\}, i \neq k$)*. Then sellers in category g will: (i) sell on Pk* ($j \in \{1, 2, ..., n\}, g, k \in \{1, 2\}$) *if* $\frac{\Theta_k}{\Theta_i} > \left[\frac{\Delta_{ig}\xi(\gamma,n)}{\widetilde{\Delta}_{kg}} \right]^2$*; and (ii) sell on Pi if* $\frac{\Theta_k}{\Theta_i} < \left[\frac{\Delta_{ig}\xi(\gamma,n)}{\widetilde{\Delta}_{kg}} \right]^2$.

<u>Proof.</u> Condition E implies that Pi will enter Sj's market if Sj sells on Pi ($j \in \{1, 2, ..., n\}, i \in \{1, 2\}$). Lemma 21 implies that Sj's profit is $\frac{\Theta_k[1-\gamma][1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{kjg}\right]^2}{4[2+\gamma(n-3)]^2}$ if Sj sells on Pk ($j \in \{1, 2, ..., n\}, k \in \{1, 2\}$). Lemma 24 implies that Sj's profit is
$\Theta_i [1 - \gamma] [1 + \gamma (n - 1)] [1 + \gamma (n - 2)] \{ \frac{\Delta_{ijg}\left\{2[1+\gamma(n-2)]^2+\gamma^2[n-1]\right\}}{\gamma^2[n-1]+4[1+\gamma(n-2)]^2[2+\gamma(n-2)]} \}^2$ if Sj sells on Pi. Therefore, Sj will sell on Pk if:

$$\frac{\Theta_k [1 - \gamma] [1 + \gamma (n - 2)] [1 + \gamma (n - 1)] \left[\widetilde{\Delta}_{kjg}\right]^2}{4 [2 + \gamma (n - 3)]^2}$$

$$> \Theta_i [1 - \gamma] [1 + \gamma (n - 1)] [1 + \gamma (n - 2)] \left\{ \frac{\Delta_{ijg} \left\{2 [1 + \gamma (n - 2)]^2 + \gamma^2 [n - 1]\right\}}{\gamma^2 [n - 1] + 4 [1 + \gamma (n - 2)]^2 [2 + \gamma (n - 2)]} \right\}^2$$

$$\Leftrightarrow \frac{\Theta_k \left[\widetilde{\Delta}_{kjg}\right]^2}{4 [2 + \gamma (n - 3)]^2} > \Theta_i \left\{ \frac{\Delta_{ijg} \left\{2 [1 + \gamma (n - 2)]^2 + \gamma^2 [n - 1]\right\}}{\gamma^2 [n - 1] + 4 [1 + \gamma (n - 2)]^2 [2 + \gamma (n - 2)]} \right\}^2$$

$$\Leftrightarrow \frac{\Theta_k}{\Theta_i} > \left\{ \frac{2\Delta_{ig} [2 + \gamma (n - 3)] \left\{2 [1 + \gamma (n - 2)]^2 + \gamma^2 [n - 1]\right\}}{\widetilde{\Delta}_{kg} \left\{\gamma^2 [n - 1] + 4 [1 + \gamma (n - 2)]^2 [2 + \gamma (n - 2)]\right\}} \right\}^2 \Leftrightarrow \frac{\Theta_k}{\Theta_i} > \left[\frac{\Delta_{ig}\xi (\gamma, n)}{\widetilde{\Delta}_{kg}} \right]^2. \blacksquare$$

Proposition 6. *Suppose Assumptions 2 and 3, and Condition E hold. When one platform is sufficiently stronger than the other platform, sellers in both categories compete with the strong platform; when one platform is reletively stronger than the other platform, sellers in category 1 compete with the strong platform whereas sellers in category 2 compete with the weak platform or face no competition from platforms; when two platforms are sufficiently similar in platform strength, sellers in both categories face no competition from platforms.*

<u>Proof.</u> Condition E ensures that each platform enters each seller's market if the platform makes no commitment. Since category 1 and category 2 sellers offer independent products, the choice of platform by category 1 sellers is independent of the choice made by category 2 sellers. (116) implies that:

$$\frac{\Delta_{21}}{\Delta_{11}} < \frac{\Delta_{22}}{\Delta_{12}} \Leftrightarrow \frac{d^{S1} - c^{S1} - \gamma [d_2 - c_2^P]}{d^{S1} - c^{S1} - \gamma [d_1 - c_1^P]} < \frac{d^{S2} - c^{S2} - \gamma [d_2 - c_2^P]}{d^{S2} - c^{S2} - \gamma [d_1 - c_1^P]}$$

$$\Leftrightarrow \left\{d^{S1} - c^{S1} - \gamma [d_2 - c_2^P]\right\} \left\{d^{S2} - c^{S2} - \gamma [d_1 - c_1^P]\right\}$$
$$< \left\{d^{S1} - c^{S1} - \gamma [d_1 - c_1^P]\right\} \left\{d^{S2} - c^{S2} - \gamma [d_2 - c_2^P]\right\}$$

$$\Leftrightarrow [d^{S1} - c^{S1}] [d^{S2} - c^{S2}] - \gamma [d_1 - c_1^P] [d^{S1} - c^{S1}] - \gamma [d_2 - c_2^P] [d^{S2} - c^{S2}] + \gamma\gamma [d_2 - c_2^P] [d_1 - c_1^P]$$
$$< [d^{S1} - c^{S1}] [d^{S2} - c^{S2}] - \gamma [d_2 - c_2^P] [d^{S1} - c^{S1}] - \gamma [d_1 - c_1^P] [d^{S2} - c^{S2}] + \gamma\gamma [d_1 - c_1^P] [d_2 - c_2^P]$$

$$\Leftrightarrow -\gamma \left[d_1 - c_1^P \right] \left[d^{S1} - c^{S1} \right] - \gamma \left[d_2 - c_2^P \right] \left[d^{S2} - c^{S2} \right]$$
$$< -\gamma \left[d_2 - c_2^P \right] \left[d^{S1} - c^{S1} \right] - \gamma \left[d_1 - c_1^P \right] \left[d^{S2} - c^{S2} \right]$$
$$\Leftrightarrow -\gamma \left[d_1 - c_1^P \right] \left[d^{S1} - c^{S1} \right] - \gamma \left[d_2 - c_2^P \right] \left[d^{S2} - c^{S2} \right]$$
$$+\gamma \left[d_2 - c_2^P \right] \left[d^{S1} - c^{S1} \right] + \gamma \left[d_1 - c_1^P \right] \left[d^{S2} - c^{S2} \right] < 0$$
$$\Leftrightarrow \gamma \left[d_2 - c_2^P \right] \left[d^{S1} - c^{S1} - \left(d^{S2} - c^{S2} \right) \right] + \gamma \left[d_1 - c_1^P \right] \left[d^{S2} - c^{S2} - \left(d^{S1} - c^{S1} \right) \right] < 0$$
$$\Leftrightarrow \gamma \left[d_2 - c_2^P \right] \left[d^{S1} - c^{S1} - \left(d^{S2} - c^{S2} \right) \right] - \gamma \left[d_1 - c_1^P \right] \left[d^{S1} - c^{S1} - \left(d^{S2} - c^{S2} \right) \right] < 0$$
$$\Leftrightarrow \gamma \left[d_2 - c_2^P - \left(d_1 - c_1^P \right) \right] \left[d^{S1} - c^{S1} - \left(d^{S2} - c^{S2} \right) \right] < 0. \tag{193}$$

(193) holds due to Assumptions 2 and 3.

(118) and (193) imply that:

$$1 < \left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2 < \left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2. \tag{194}$$

(86) and (116) imply that for $i, g \in \{1, 2\}$:

$$\frac{\Delta_{ig}}{\widetilde{\Delta}_{kg}} = \frac{d^{Sg} - c^{Sg} - \gamma \left[d_i - c_i^P \right]}{\left[1 - \gamma \right] \left[1 + \gamma \left(n - 1 \right) \right]} \frac{1 + \gamma \left[n - 1 \right]}{d^{Sg} - c^{Sg}} = \frac{d^{Sg} - c^{Sg} - \gamma \left[d_i - c_i^P \right]}{\left[1 - \gamma \right] \left[d^{Sg} - c^{Sg} \right]}$$
$$= \frac{d^{Sg} - c^{Sg} - \gamma \left[d_i - c_i^P \right]}{d^{Sg} - c^{Sg} - \gamma \left[d^{Sg} - c^{Sg} \right]} < 1. \tag{195}$$

(195) holds because of (88).

(192) and (195) imply that for $i, g \in \{1, 2\}$:

$$\left[\frac{\Delta_{ig} \xi \left(\gamma, n \right)}{\widetilde{\Delta}_{kg}} \right]^2 < 1 \text{ and } \left[\frac{\widetilde{\Delta}_{kg}}{\Delta_{ig} \xi \left(\gamma, n \right)} \right]^2 > 1. \tag{196}$$

(86) and (116) imply that:

$$\frac{\Delta_{11}}{\widetilde{\Delta}_{21}} > \frac{\Delta_{12}}{\widetilde{\Delta}_{22}} \Leftrightarrow \frac{d^{S1} - c^{S1} - \gamma \left[d_1 - c_1^P \right]}{d^{S1} - c^{S1} - \gamma \left[d^{S1} - c^{S1} \right]} > \frac{d^{S2} - c^{S2} - \gamma \left[d_1 - c_1^P \right]}{d^{S2} - c^{S2} - \gamma \left[d^{S2} - c^{S2} \right]}$$
$$\Leftrightarrow \frac{d^{S1} - c^{S1} - \gamma \left[d_1 - c_1^P \right]}{\left[1 - \gamma \right] \left[d^{S1} - c^{S1} \right]} > \frac{d^{S2} - c^{S2} - \gamma \left[d_1 - c_1^P \right]}{\left[1 - \gamma \right] \left[d^{S2} - c^{S2} \right]}$$
$$\Leftrightarrow \frac{d^{S1} - c^{S1} - \gamma \left[d_1 - c_1^P \right]}{d^{S1} - c^{S1}} > \frac{d^{S2} - c^{S2} - \gamma \left[d_1 - c_1^P \right]}{d^{S2} - c^{S2}}$$
$$\Leftrightarrow 1 - \frac{\gamma \left[d_1 - c_1^P \right]}{d^{S1} - c^{S1}} > 1 - \frac{\gamma \left[d_1 - c_1^P \right]}{d^{S2} - c^{S2}} \Leftrightarrow \frac{d_1 - c_1^P}{d^{S1} - c^{S1}} < \frac{d_1 - c_1^P}{d^{S2} - c^{S2}} \Leftrightarrow d^{S2} - c^{S2} < d^{S1} - c^{S1}. \tag{197}$$

The last inequality in (197) holds because of (88).

(196) and (197) imply that:

$$1 < \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2 < \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2. \tag{198}$$

(86) and (116) imply that:

$$\begin{aligned}
&\frac{\Delta_{21}}{\Delta_{11}} < \frac{\widetilde{\Delta}_{21}}{\Delta_{11}} \Leftrightarrow \frac{\Delta_{21}}{\widetilde{\Delta}_{21}} < 1 \Leftrightarrow \frac{d^{S1} - c^{S1} - \gamma\left[d_2 - c_2^P\right]}{d^{S1} - c^{S1} - \gamma\left[d^{S1} - c^{S1}\right]} < 1 \\
&\Leftrightarrow d^{S1} - c^{S1} - \gamma\left[d_2 - c_2^P\right] < d^{S1} - c^{S1} - \gamma\left[d^{S1} - c^{S1}\right] \\
&\Leftrightarrow -\gamma\left[d_2 - c_2^P\right] < -\gamma\left[d^{S1} - c^{S1}\right] \Leftrightarrow d_2 - c_2^P > d^{S1} - c^{S1}; \qquad (199) \\
&\frac{\Delta_{22}}{\Delta_{12}} < \frac{\widetilde{\Delta}_{22}}{\Delta_{12}} \Leftrightarrow \frac{\Delta_{22}}{\widetilde{\Delta}_{22}} < 1 \Leftrightarrow \frac{d^{S2} - c^{S2} - \gamma\left[d_2 - c_2^P\right]}{d^{S2} - c^{S2} - \gamma\left[d^{S2} - c^{S2}\right]} < 1 \\
&\Leftrightarrow d^{S2} - c^{S2} - \gamma\left[d_2 - c_2^P\right] < d^{S2} - c^{S2} - \gamma\left[d^{S2} - c^{S2}\right] \\
&\Leftrightarrow -\gamma\left[d_2 - c_2^P\right] < -\gamma\left[d^{S2} - c^{S2}\right] \Leftrightarrow d_2 - c_2^P > d^{S2} - c^{S2}. \qquad (200)
\end{aligned}$$

The last inequalities in (199) and (200) hold because of (88).

(192) and (199) imply that:

$$\left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2 < \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2. \tag{201}$$

(192) and (200) imply that:

$$\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 < \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2. \tag{202}$$

(194) and (202) imply that:

$$1 < \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2 < \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 < \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2. \tag{203}$$

Case I. $\Theta_1 > \Theta_2$.

Lemma 25 implies that if both platforms commit not to enter, then sellers in both categories 1 and 2 will sell on P1 because $\Theta_1 > \Theta_2$.

Lemma 27 implies that if P1 commits not to enter and P2 makes no commitment, then sellers in categories 1 and 2 will sell on P1 because $\frac{\Theta_1}{\Theta_2} > 1$ by assumption and $\left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}}\right]^2 < 1$ and $\left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}}\right]^2 < 1$ from (196).

Case I(i). $\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 < \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2$.

(198) and (203) imply that:

$$1<\left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2<\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2<\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi\left(\gamma,n\right)}\right]^2<\left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi\left(\gamma,n\right)}\right]^2. \tag{204}$$

First suppose $\frac{\Theta_1}{\Theta_2}>\left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2$. Lemmas 26 and (204) imply that sellers in both categories 1 and 2 sell on P1, if platforms both make no commitment. Lemma 27, (192) and (204) imply that sellers in both categories 1 and 2 sell on P1, if P2 commits not to enter and P1 makes no commitment, because

$$\frac{\Theta_2}{\Theta_1}<\left[\frac{\Delta_{11}\xi\left(\gamma,n\right)}{\widetilde{\Delta}_{21}}\right]^2\Leftrightarrow\frac{\Theta_1}{\Theta_2}>\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi\left(\gamma,n\right)}\right]^2\text{, and} \tag{205}$$

$$\frac{\Theta_2}{\Theta_1}<\left[\frac{\Delta_{12}\xi\left(\gamma,n\right)}{\widetilde{\Delta}_{22}}\right]^2\Leftrightarrow\frac{\Theta_1}{\Theta_2}>\left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi\left(\gamma,n\right)}\right]^2. \tag{206}$$

The last inequalities in (205) and (206) hold because of (88). Consequently, sellers in both categories 1 and 2 sell on P1, regardless of platforms' commitments. Condition E ensures P1 enters each seller's market if P1 makes no commitment. Therefore, Lemmas 21 and 24 imply that P1's profit is: (i) $\Theta_1H_{11}-F+\Theta_1H_{12}-F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{11}\right)^2+\left(\widetilde{\Delta}_{12}\right)^2\right]}{4[2+\gamma(n-3)]}$ if P1 commits not to enter. Condition E ensures that $\Theta_1H_{1g}-F>\frac{\Theta_1n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\widetilde{\Delta}_{1g}\right]^2}{4[2+\gamma(n-3)]}$ for $g\in\{1,2\}$. Therefore, in equilibrium, P1 makes no commitment, and sellers in both categories 1 and 2 sell on P1 if $\frac{\Theta_1}{\Theta_2}>\left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2$.

Next suppose $\frac{\Theta_1}{\Theta_2}\in\left(\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2,\left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2\right)$. Lemmas 26 and (204) imply that sellers in both categories 1 and 2 sell on P1, if platforms both make no commitment. Lemma 27, (192) and (204) imply that if P2 commits not to enter and P1 makes no commitment, then sellers in category 1 sell on P1 and sellers in category 2 sell on P2, because

$$\frac{\Theta_2}{\Theta_1}<\left[\frac{\Delta_{11}\xi\left(\gamma,n\right)}{\widetilde{\Delta}_{21}}\right]^2\Leftrightarrow\frac{\Theta_1}{\Theta_2}>\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi\left(\gamma,n\right)}\right]^2\text{, and} \tag{207}$$

$$\frac{\Theta_2}{\Theta_1}>\left[\frac{\Delta_{12}\xi\left(\gamma,n\right)}{\widetilde{\Delta}_{22}}\right]^2\Leftrightarrow\frac{\Theta_1}{\Theta_2}<\left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi\left(\gamma,n\right)}\right]^2. \tag{208}$$

The last inequalities in (207) and (208) hold because of (88). Consequently, sellers in category 1 sell on P1, regardless of the platforms' commitments. If P2 makes no commitment, sellers in category 2 sell on P1. If P2 commits not to enter, sellers in category 2: (i) sell on P1 if P1 commits not to enter; and (ii) sell on P2 if P1 makes no commitment. Lemmas 21 and 24 imply that P1's profit is: (i) $\Theta_1H_{11}-F$ if P1 makes no commitment; and (ii)

$\frac{\Theta_1 n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{11}\right)^2+\left(\widetilde{\Delta}_{12}\right)^2\right]}{4[2+\gamma(n-3)]}$ if P1 commits not to enter. Condition E ensures that P1 secures more profit by making no commitment than by committing not to enter. Therefore, in equilibrium, P1 makes no commitment whereas P2 commits not to enter, and sellers in category 1 sell on P1 whereas sellers in category 2 sell on P2, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2, \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2 \right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2, \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2 \right)$. Lemmas 26 and (204) imply that sellers in both categories 1 and 2 sell on P1, if platforms both make no commitment. Lemma 27, (192) and (204) imply that if P2 commits not to enter and P1 makes no commitment, then sellers in categories 1 and 2 sell on P2, because

$$\frac{\Theta_2}{\Theta_1} > \left[\frac{\Delta_{11}\xi(\gamma,n)}{\widetilde{\Delta}_{21}}\right]^2 \Leftrightarrow \frac{\Theta_1}{\Theta_2} < \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2, \text{ and} \tag{209}$$

$$\frac{\Theta_2}{\Theta_1} > \left[\frac{\Delta_{12}\xi(\gamma,n)}{\widetilde{\Delta}_{22}}\right]^2 \Leftrightarrow \frac{\Theta_1}{\Theta_2} < \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2. \tag{210}$$

The last inequalities in (209) and (210) hold because of (88). Consequently, if P2 makes no commitment, sellers in both categories 1 and 2 sell on P1. If P2 commits not to enter, sellers in both categories 1 and 2: (i) sell on P1 if P1 commits not to enter; and (ii) sell on P2 if P1 makes no commitment. Therefore, in equilibrium, both P1 and P2 commit not to enter, and sellers in both categories 1 and 2 sell on P1, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2, \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2 \right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 \right)$. Lemmas 26 and (204) imply that sellers in category 1 sell on P1 whereas sellers in category 2 sell on P2, if platforms both make no commitment. Lemma 27, (192) and (204) imply that if P2 commits not to enter and P1 makes no commitment, then sellers in categories 1 and 2 sell on P2, because (209) and (210) hold in this case. Consequently, if P2 makes no commitment, Lemmas 21 and 24 imply that P1's profit is: (i) $\Theta_1 H_{11} - F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1 n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{11}\right)^2+\left(\widetilde{\Delta}_{12}\right)^2\right]}{4[2+\gamma(n-3)]}$ if P1 commits not to enter. Condition E ensures that P1 secures more profit by making no commitment than by committing not to enter if P2 makes no commitment. If P2 commits not to enter, P1 secures more profit by committing not to enter than by making no commitment. If P1 makes no commitment, Lemmas 21 and 24 imply that P2's profit is: (i) $\Theta_2 H_{22} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{21}\right)^2+\left(\widetilde{\Delta}_{22}\right)^2\right]}{4[2+\gamma(n-3)]}$ if P2 commits not to enter. Condition E ensures that P2 secures more profit by committing not to enter than by making no commitment. If P1 commits not to enter, P2 is indifferent between committing not to enter and making no commitment. Therefore, in equilibrium, both P1 and P2 commit not to enter, and both sellers in categories 1 and 2 sell on P1, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 \right)$.

Finally, suppose $\frac{\Theta_1}{\Theta_2} \in \left(1, \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2 \right)$. Lemmas 26 and (204) imply that sellers in both categories 1 and 2 sell on P2, if platforms both make no commitment. Lemma 27, (192) and (204) imply that if P2 commits not to enter and P1 makes no commitment, then sellers in categories 1 and 2 sell on P2, because (209) and (210) hold in this case. Consequently, if P1 makes no commitment, then sellers in both categories 1 and 2 sell on P2. Therefore, in equilibrium, P1 commits not to enter, and sellers in both categories sell on P1, if $\frac{\Theta_1}{\Theta_2} \in \left(1, \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2\right)$.

Case I(ii). $\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 > \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2$.

(198) and (201) imply that:

$$1 < \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2 < \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi\left(\gamma,n\right)}\right]^2 < \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 < \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi\left(\gamma,n\right)}\right]^2. \tag{211}$$

First suppose $\frac{\Theta_1}{\Theta_2} > \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2$. Then, in equilibrium, P1 makes no commitment, and sellers in both categories 1 and 2 sell on P1. The analysis is analogous to that under Case I when $\frac{\Theta_1}{\Theta_2} > \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2, \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2\right)$. Then, in equilibrium, P1 makes no commitment whereas P2 commits not to enter, and sellers in category 1 sell on P1 whereas sellers in category 2 sell on P2. The analysis is analogous to that under Case I when $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2, \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2\right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2\right)$. Lemmas 26 and (211) imply that sellers in category 1 sell on P1 and sellers in category 2 sell on P2, if platforms both make no commitment. Lemma 27, (192) and (211) imply that if P2 commits not to enter and P1 makes no commitment, then sellers in category 1 sell on P1 and sellers in category 2 sell on P2, because (207) and (208) hold. Consequently, if P1 makes no commitment, then sellers in category 1 sell on P1 and sellers in category 2 sell on P2. If P1 commits not to enter, then sellers in both categories 1 and 2 sell on P1. Lemmas 21 and 24 imply that P1's profit is: (i) $\Theta_1 H_{11} - F$ if P1 makes no commitment; and (ii) $\frac{\Theta_1 n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{11}\right)^2+\left(\widetilde{\Delta}_{12}\right)^2\right]}{4[2+\gamma(n-3)]}$ if P1 commits not to enter. Condition E ensures that P1 secures more profit by making no commitment than by committing not to enter in this case. Therefore, in equilibrium, platforms both make no commitment, and sellers in category 1 sell on P1 and sellers in category 2 sell on P2, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2\right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2, \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2\right)$. Then, in equilibrium, both P1 and P2 commit not to enter, and both sellers in categories 1 and 2 sell on P1. The analysis is analogous to that under Case I when $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2, \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2\right)$.

Finally suppose $\frac{\Theta_1}{\Theta_2} \in \left(1, \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2\right)$. Then, in equilibrium, P1 commits not to enter, and sellers in both categories sell on P1. The analysis is analogous to that under Case I when $\frac{\Theta_1}{\Theta_2} \in \left(1, \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2\right)$.

Case II. $\Theta_1 < \Theta_2$.

Lemma 25 implies that if both platforms commit not to enter, then sellers in both categories 1 and 2 will sell on P2 because $\Theta_1 < \Theta_2$.

Lemma 27 implies that if P2 commits not to enter and P1 makes no commitment, then sellers in categories 1 and 2 will sell on P2 because $\frac{\Theta_2}{\Theta_1} > 1$ and $\left[\frac{\Delta_{11}\xi(\gamma,n)}{\widetilde{\Delta}_{21}}\right]^2 < 1$ and $\left[\frac{\Delta_{12}\xi(\gamma,n)}{\widetilde{\Delta}_{22}}\right]^2 < 1$ from (196).

Lemma 26 implies that if platforms both make no commitment, then sellers in both categories 1 and 2 will sell on P2 because $\frac{\Theta_1}{\Theta_2} < 1$ and $\left[\frac{\Delta_{2g}}{\Delta_{1g}}\right]^2 > 1$ ($g \in \{1, 2\}$) from (204).

(86) and (116) imply that:

$$\frac{\Delta_{21}}{\widetilde{\Delta}_{11}} > \frac{\Delta_{22}}{\widetilde{\Delta}_{12}} \Leftrightarrow \frac{d^{S1} - c^{S1} - \gamma\left[d_2 - c_2^P\right]}{d^{S1} - c^{S1}} > \frac{d^{S2} - c^{S2} - \gamma\left[d_2 - c_2^P\right]}{d^{S2} - c^{S2}}$$

$$\Leftrightarrow 1 - \frac{\gamma\left[d_2 - c_2^P\right]}{d^{S1} - c^{S1}} > 1 - \frac{\gamma\left[d_2 - c_2^P\right]}{d^{S2} - c^{S2}} \Leftrightarrow \frac{\gamma\left[d_2 - c_2^P\right]}{d^{S1} - c^{S1}} < \frac{\gamma\left[d_2 - c_2^P\right]}{d^{S2} - c^{S2}} \Leftrightarrow d^{S2} - c^{S2} < d^{S1} - c^{S1}. \tag{212}$$

(192), (196), and (212) imply that:

$$\left[\frac{\Delta_{22}\xi(\gamma, n)}{\widetilde{\Delta}_{12}}\right]^2 < \left[\frac{\Delta_{21}\xi(\gamma, n)}{\widetilde{\Delta}_{11}}\right]^2 < 1. \tag{213}$$

First suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}}\right]^2, 1\right)$. Lemma 27 implies that if P1 commits not to enter and P2 makes no commitment, then sellers in both categories sell on P1. Therefore, in equilibrium, both P1 and P2 commit not to enter, and sellers in both categories 1 and 2 will sell on P2 if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}}\right]^2, 1\right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}}\right]^2, \left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}}\right]^2\right)$. Lemma 27 implies that if P1 commits not to enter and P2 makes no commitment, then sellers in category 2 sell on P1 whereas sellers in category 1 sell on P2. Therefore, if P2 commits not to enter, sellers in both categories sell on P2, regardless of P1's commitment. If P2 makes no commitment, then P1 commits not to enter, and sellers in category 2 sell on P1 whereas sellers in category 1 sell on P2. Lemmas 21 and 24 imply that P2's profit is: (i) $\Theta_2 H_{21} - F$ if P2 makes no commitment; and (ii) $\frac{\Theta_2 n[1+\gamma(n-2)][1+\gamma(n-1)]\left[\left(\widetilde{\Delta}_{21}\right)^2 + \left(\widetilde{\Delta}_{22}\right)^2\right]}{4[2+\gamma(n-3)]}$ if P2 commits not to enter. Condition E ensures that P2 secures more profit by making no commitment than by committing not to enter in this case. Consequently, in equilibrium, P2 makes no commitment whereas P1 commits not to enter, and sellers in category 2 sell on P1 whereas sellers in category 1 sell on P2 if

$\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}} \right]^2, \left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}} \right]^2 \right)$.

Finally suppose $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}} \right]^2$. Lemma 27 implies that if P1 commits not to enter and P2 makes no commitment, then sellers in both categories sell on P2. Therefore, sellers in both categories sell on P2, regardless of platforms' commitments. Consequently, in equilibrium, P2 makes no commitment, and sellers in both categories sell on P2 if $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}} \right]^2$.

In summary, when one platform is sufficiently stronger than the other platform (i.e., $\frac{\Theta_1}{\Theta_2} > \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)} \right]$ or $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}} \right]^2$), sellers in both categories compete with the strong platform; when one platform is reletively stronger than the other platform (i.e., $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)} \right]^2, \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)} \right]^2 \right)$ or $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}} \right]^2, \left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}} \right]^2 \right)$), sellers in category 1 compete with the strong platform whereas sellers in category 2 compete with the weak platform or face no competition from platforms; when two platforms are sufficiently similar in platform strength (i.e., $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}} \right]^2, \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)} \right]^2 \right)$), sellers in both categories face no competition from platforms. ■

Proposition 7. *Suppose Assumptions 2 and 3, and Condition E hold. Then increased platform competition reduces platform-seller competition in the sense that at least sellers in one category face no competition or reduced competition from platforms in the presence of platform competition.*

Proof. Proposition 5 shows that sellers in both categories compete with the monopoly platform (e.g., P1) in equilibrium.
Case I. The platform in the monopolistic setting faces a weaker platform in the competition setting (i.e., $\Theta_1 > \Theta_2$).

Proposition 6 shows that: (i) when P1 is sufficiently stronger than P2, sellers in both categories compete with P1; (ii) when P1 is reletively stronger than P2, sellers in category 1 compete with P1 whereas sellers in category 2 compete with P2 or face no competition from platforms; and (iii) when P1 is sufficiently similar to P2, sellers in both categories face no competition from platforms.
Case II. The platform in the monopolistic setting faces a stronger platform in the competition setting (i.e., $\Theta_1 < \Theta_2$).

Proposition 6 shows that: (i) when P2 is sufficiently stronger than P1, sellers in both categories compete with P2; (ii) when P2 is reletively stronger than P1, sellers in category 1 compete with P2 whereas sellers in category 2 compete with P1 or face no competition from platforms; and (iii) when P2 is sufficiently similar to P1, sellers in both categories face no competition from platforms.

Compared to the monopolistic platform setting, increased platform competition reduces downstream seller competition in the sense that at least sellers in one category face no competition or reduced competition from platforms in the presence of platform competition. This is the case because assumption 3 ensures that P2 is a weaker seller than P1. ■

Lemma 28. *Suppose Assumptions 2 and 3, and Condition E hold. Then consumer surplus* *under MP is given by (220) and consumer surplus under PC is given by (221) and (222).*

Proof. In the presence of platform entry, the consumer surplus when consuming the product of Pk and sellers in category g is given by:

$$CS_{kg}^{E} = d_k Q_{kg}^{P-E} + d^{Sg}\left[n-1\right] Q_{kg}^{S-E} - P_{kg}^{P-E} Q_{kg}^{P-E} - \left[n-1\right] P_{kg}^{S-E} Q_{kg}^{S-E}$$

$$-\frac{1}{2}\left\{\left[Q_{kg}^{P-E}\right]^2 + \left[n-1\right]\left[Q_{kg}^{S-E}\right]^2 + 2\,\gamma\left[\left(n-1\right) Q_{kg}^{P-E} Q_{kg}^{S-E} + \left(\sum_{k=2}^{n-1}\left(n-k\right)\right)\left(Q_{kg}^{S-E}\right)^2\right]\right\}$$

$$= \left[d_k - P_{kg}^{P-E}\right] Q_{kg}^{P-E} + \left[n-1\right]\left[d^{Sg} - P_{kg}^{S-E}\right] Q_{kg}^{S-E}$$

$$-\frac{1}{2}\left\{\left[Q_{kg}^{P-E}\right]^2 + \left[n-1\right]\left[Q_{kg}^{S-E}\right]^2 + 2\,\gamma\left[\left(n-1\right) Q_{kg}^{P-E} Q_{kg}^{S-E} + \frac{\left(n-1\right)\left(n-2\right)}{2}\left(Q_{kg}^{S-E}\right)^2\right]\right\}$$

$$= \left[d_k - P_{kg}^{P-E}\right] Q_{kg}^{P-E} + \left[n-1\right]\left[d^{Sg} - P_{kg}^{S-E}\right] Q_{kg}^{S-E}$$

$$-\frac{1}{2}\left[\left(Q_{kg}^{P-E}\right)^2 + \left(n-1\right)\left(Q_{kg}^{S-E}\right)^2 + 2\,\gamma\left(n-1\right) Q_{kg}^{P-E} Q_{kg}^{S-E} + \gamma\left(n-1\right)\left(n-2\right)\left(Q_{kg}^{S-E}\right)^2\right]$$

$$= \left[d_k - P_{kg}^{P-E}\right] Q_{kg}^{P-E} + \left[n-1\right]\left[d^{Sg} - P_{kg}^{S-E}\right] Q_{kg}^{S-E}$$

$$-\frac{1}{2}\left\{\left[Q_{kg}^{P-E}\right]^2 + \left[n-1\right]\left[1+\gamma\left(n-2\right)\right]\left[Q_{kg}^{S-E}\right]^2 + 2\,\gamma\left[n-1\right] Q_{kg}^{P-E} Q_{kg}^{S-E}\right\}. \tag{214}$$

(79) implies that:

$$d_k - P_{kg}^{P-E} = Q_{kg}^{P-E} + \gamma\left[n-1\right] Q_{kg}^{S-E}; \text{ and}$$

$$d^{Sg} - P_{kg}^{S-E} = Q_{kg}^{S-E} + \gamma\left[n-2\right] Q_{kg}^{S-E} + \gamma Q_{kg}^{P-E}. \tag{215}$$

(214) and (215) imply that:

$$CS_{kg}^{E} = \left[Q_{kg}^{P-E} + \gamma\left(n-1\right) Q_{kg}^{S-E}\right] Q_{kg}^{P-E} + \left[n-1\right]\left[Q_{kg}^{S-E} + \gamma\left(n-2\right) Q_{kg}^{S-E} + \gamma Q_{kg}^{P-E}\right] Q_{kg}^{S-E}$$

$$-\frac{1}{2}\left\{\left[Q_{kg}^{P-E}\right]^2 + \left[n-1\right]\left[1+\gamma\left(n-2\right)\right]\left[Q_{kg}^{S-E}\right]^2 + 2\,\gamma\left[n-1\right] Q_{kg}^{P-E} Q_{kg}^{S-E}\right\}$$

$$= \left[Q_{kg}^{P-E}\right]^2 + \gamma\left[n-1\right] Q_{kg}^{P-E} Q_{kg}^{S-E} + \left[n-1\right]\left[Q_{kg}^{S-E}\right]^2 + \gamma\left[n-1\right]\left[n-2\right]\left[Q_{kg}^{S-E}\right]^2 + \gamma\left[n-1\right] Q_{kg}^{P-E} Q_{kg}^{S-E}$$

$$-\frac{1}{2}\left[Q_{kg}^{P-E}\right]^2 - \frac{1}{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]\left[Q_{kg}^{S-E}\right]^2 - \gamma\left[n-1\right] Q_{kg}^{P-E} Q_{kg}^{S-E}$$

$$= \frac{1}{2}\left[Q_{kg}^{P-E}\right]^2 + \frac{1}{2}\left[n-1\right]\left[1+\gamma\left(n-2\right)\right]\left[Q_{kg}^{S-E}\right]^2 + \gamma\left[n-1\right] Q_{kg}^{P-E} Q_{kg}^{S-E}. \tag{216}$$

In the absence of platform entry, the consumer surplus when consuming the product of sellers in category g on platform Pk is given by:

$$CS_{kg}^{NE} = n d^{Sg} Q_{kg}^{S-NE} - \frac{1}{2}\left[n\left(Q_{kg}^{S-NE}\right)^2 + 2\,\gamma\sum_{k=1}^{n-1}\left(n-k\right)\left(Q_{kg}^{S-NE}\right)^2\right] - n P_{kg}^{S-NE} Q_{kg}^{S-NE}$$

$$= nd^{Sg}Q_{kg}^{S-NE} - \frac{1}{2}\left[n\left(Q_{kg}^{S-NE}\right)^2 + 2\,\gamma\frac{n\,(n-1)}{2}\left(Q_{kg}^{S-NE}\right)^2\right] - nP_{kg}^{S-NE}Q_{kg}^{S-NE}$$

$$= n\left[d^{Sg} - P_{kg}^{S-NE}\right]Q_{kg}^{S-NE} - \frac{n}{2}\left[Q_{kg}^{S-NE}\right]^2 - \frac{\gamma n\,[n-1]}{2}\left[Q_{kg}^{S-NE}\right]^2$$

$$= n\left[d^{Sg} - P_{kg}^{S-NE}\right]Q_{kg}^{S-NE} - \frac{n\,[1+\gamma\,(n-1)]}{2}\left[Q_{kg}^{S-NE}\right]^2. \tag{217}$$

(79) implies that:

$$d^{Sg} - P_{kg}^{S-NE} = Q_{kg}^{S-NE} + \gamma\,[n-1]\,Q_{kg}^{S-NE} = [1+\gamma\,(n-1)]\,Q_{kg}^{S-NE}. \tag{218}$$

(214) and (215) imply that:

$$CS_{kg}^{NE} = n\,[1+\gamma\,(n-1)]\left[Q_{kg}^{S-NE}\right]^2 - \frac{n\,[1+\gamma\,(n-1)]}{2}\left[Q_{kg}^{S-NE}\right]^2 = \frac{n\,[1+\gamma\,(n-1)]}{2}\left[Q_{kg}^{S-NE}\right]^2. \tag{219}$$

Proposition 5 shows that when P1 is the monopoly platform, then (216) implies that the consumer surplus is:

$$CS^{MP} = CS_{11}^{E} + CS_{12}^{E}. \tag{220}$$

<u>Case I.</u> $\Theta_1 > \Theta_2$.

Proposition 6 shows that when P1 competes with P2 and: (i) $\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 < \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2$, the consumer surplus is:

$$CS^{PC} = \begin{cases} CS_{11}^{E} + CS_{12}^{E}, & \text{if } \frac{\Theta_1}{\Theta_2} > \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2 \\ CS_{11}^{E} + CS_{22}^{NE}, & \text{if } \frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2, \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2\right) \\ CS_{11}^{NE} + CS_{12}^{NE}, & \text{if } \frac{\Theta_1}{\Theta_2} \in \left(1, \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2\right) \end{cases};$$

and (ii) $\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 > \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2$, the consumer surplus is:

$$CS^{PC} = \begin{cases} CS_{11}^{E} + CS_{12}^{E}, & \text{if } \frac{\Theta_1}{\Theta_2} > \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2 \\ CS_{11}^{E} + CS_{22}^{NE}, & \text{if } \frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2, \left[\frac{\widetilde{\Delta}_{22}}{\Delta_{12}\xi(\gamma,n)}\right]^2\right) \\ CS_{11}^{E} + CS_{22}^{E}, & \text{if } \frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2\right) \\ CS_{11}^{NE} + CS_{12}^{NE}, & \text{if } \frac{\Theta_1}{\Theta_2} \in \left(1, \left[\frac{\widetilde{\Delta}_{21}}{\Delta_{11}\xi(\gamma,n)}\right]^2\right) \end{cases}. \tag{221}$$

<u>Case II.</u> $\Theta_1 < \Theta_2$.

Proposition 6 shows that when P1 competes with P2, the comsumer surplus is:

$$CS^{PC} = \begin{cases} CS_{21}^{NE} + CS_{22}^{NE}, & \text{if } \frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}} \right]^2, 1 \right) \\ CS_{12}^{NE} + CS_{21}^{E}, & \text{if } \frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}} \right]^2, \left[\frac{\Delta_{21}\xi(\gamma,n)}{\widetilde{\Delta}_{11}} \right]^2 \right) \\ CS_{21}^{E} + CS_{22}^{E}, & \text{if } \frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{22}\xi(\gamma,n)}{\widetilde{\Delta}_{12}} \right]^2 \end{cases} . \blacksquare \qquad (222)$$

Figure 1 illustrates numerical solutions for settings where $d^{S2} - c^{S2} = 5$, $d^{S1} - c^{S1} = 6$, $d_2 - c_2^P = 7$, $d_1 - c_1^P = 8$, $\gamma = 0.5$, $\frac{\Theta_1}{\Theta_2} \in [0.1, 10]$, and $n \in [2, 100]$. The X-axis represents the number of sellers in each category and the Y-axis represents the platform strength ratio. Figure 1 demonstrates that when seller entry is taken into account, increased platform competition can reduce consumer surplus unless the competing platform's platform strength is sufficiently pronounced. Figure 1 also indicates that increased seller competition slightly reduces this effect.

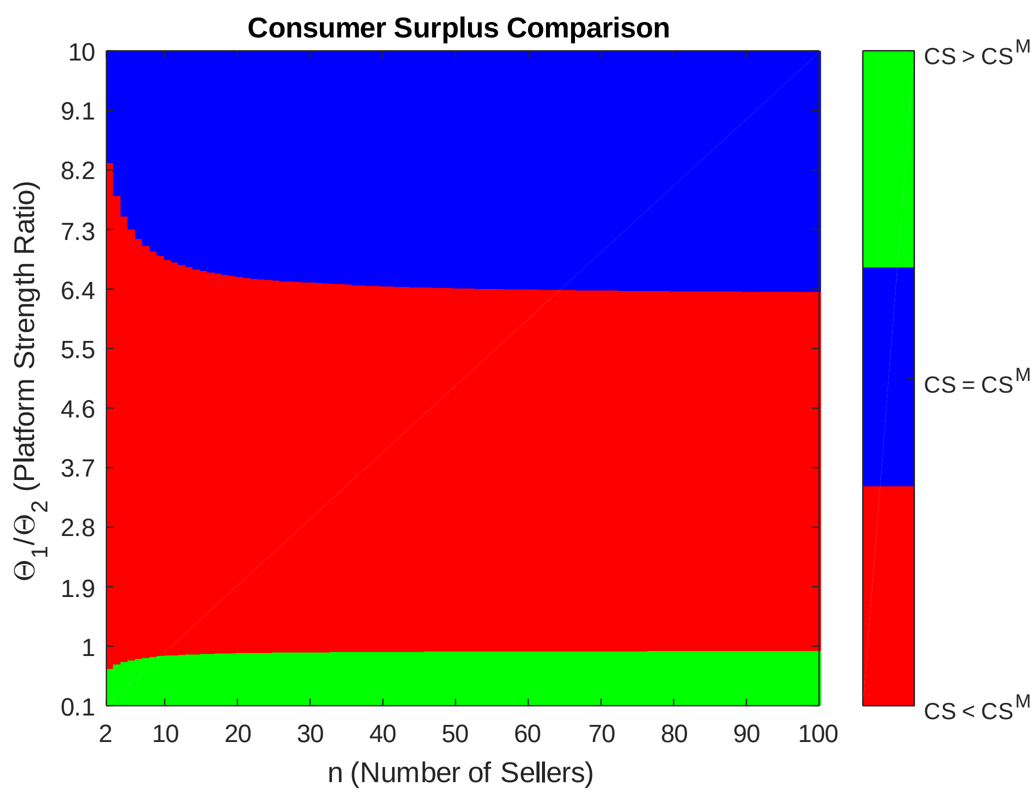

Figure 1: Comparison between CS and CS^M in the setting of seller competition

Figure 2 illustrates numerical solutions for settings where $d^{S2} - c^{S2} = 5$, $d^{S1} - c^{S1} = 6$, $d_2 - c_2^P = 7$, $d_1 - c_1^P = 8$, $\gamma = 0.7$, $\frac{\Theta_1}{\Theta_2} \in [0.1, 10]$, and $n \in [2, 100]$.

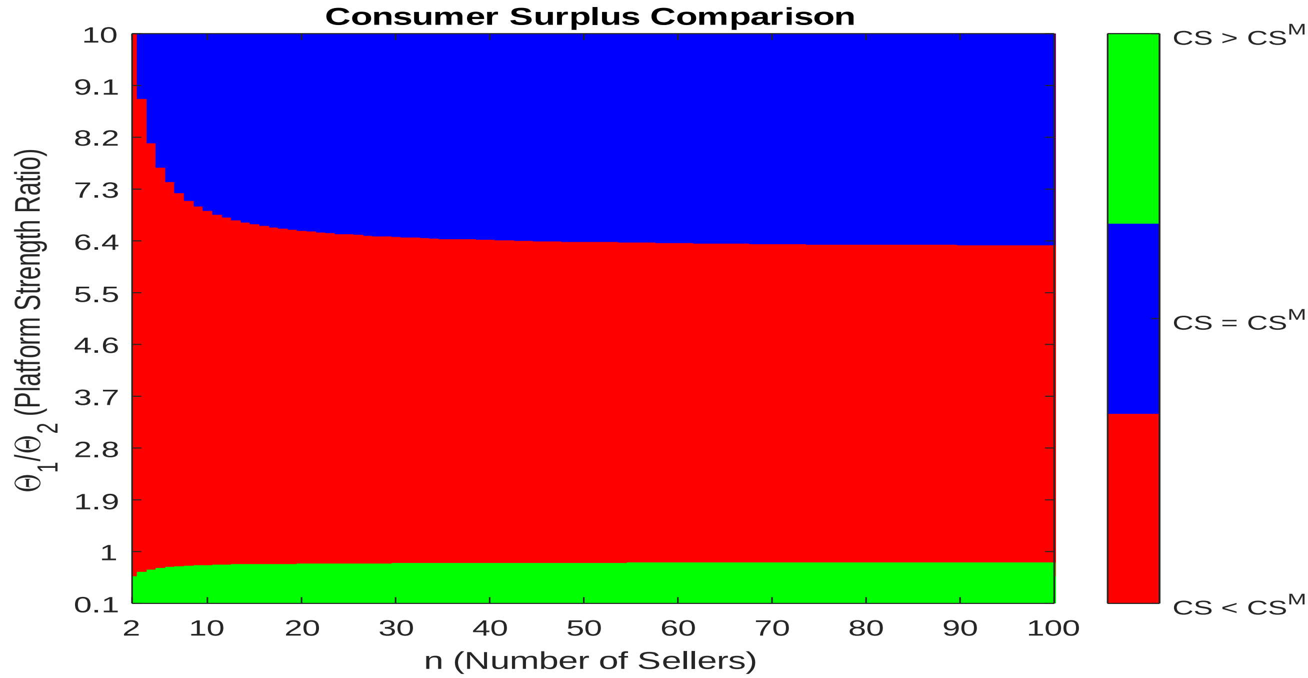

Figure 2: Comparison between CS and CS^M when $\gamma = 0.7$

Figure 3 illustrates numerical solutions for settings where $d^{S2} - c^{S2} = 5$, $d^{S1} - c^{S1} = 6$, $d_2 - c_2^P = 7$, $d_1 - c_1^P = 8$, $\gamma = 0.3$, $\frac{\Theta_1}{\Theta_2} \in [0.1, 10]$, and $n \in [2, 100]$.

3 Endogenous platform strength

In this section, I consider the case where platform strength is endogenised. I consider a game in which Pi chooses whether to invest K to increase its platform strength from Θ_L to Θ_H ($\Theta_L < \Theta_H$). Then platforms simultaneously choose to either commit not to act as sellers or make no such commitment. After the platform strengths and commitments are specified, S1 and S2 choose the platform on which they will sell (simultaneously and independently). Next, a platform that made no commitment will make its entry decision (i.e., whether to enter and which market to enter). Then P1 and P2 simultaneously set their per-unit commissions. Finally, each active seller sets its profit-maximizing price for its product.

Condition 5. $K < \min\{\Theta_H\,[M_{k1} + M_{k2}] - 2F - \frac{\Theta_L\left[\widetilde{\Delta}_{kj}\right]^2}{8\,b_j^S}, \frac{\Theta_H\left[\widetilde{\Delta}_{k2}\right]^2}{8\,b_2^S} - \frac{\Theta_L\left[\widetilde{\Delta}_{k2}\right]^2}{8\,b_2^S}, \Theta_H\,M_{k1} - F - \frac{\Theta_L\left[\widetilde{\Delta}_{k1}\right]^2}{8\,b_1^S}\}$.

Proposition 8. *Suppose strength enhancement is feasible (Condition 5 holds). In the platform competition setting, in equilibrium, both P1 and P2 invest to increase its platform strength from Θ_L to Θ_H, both platforms commit not to enter, and each seller is indifferent between selling on P1 and selling on P2.*

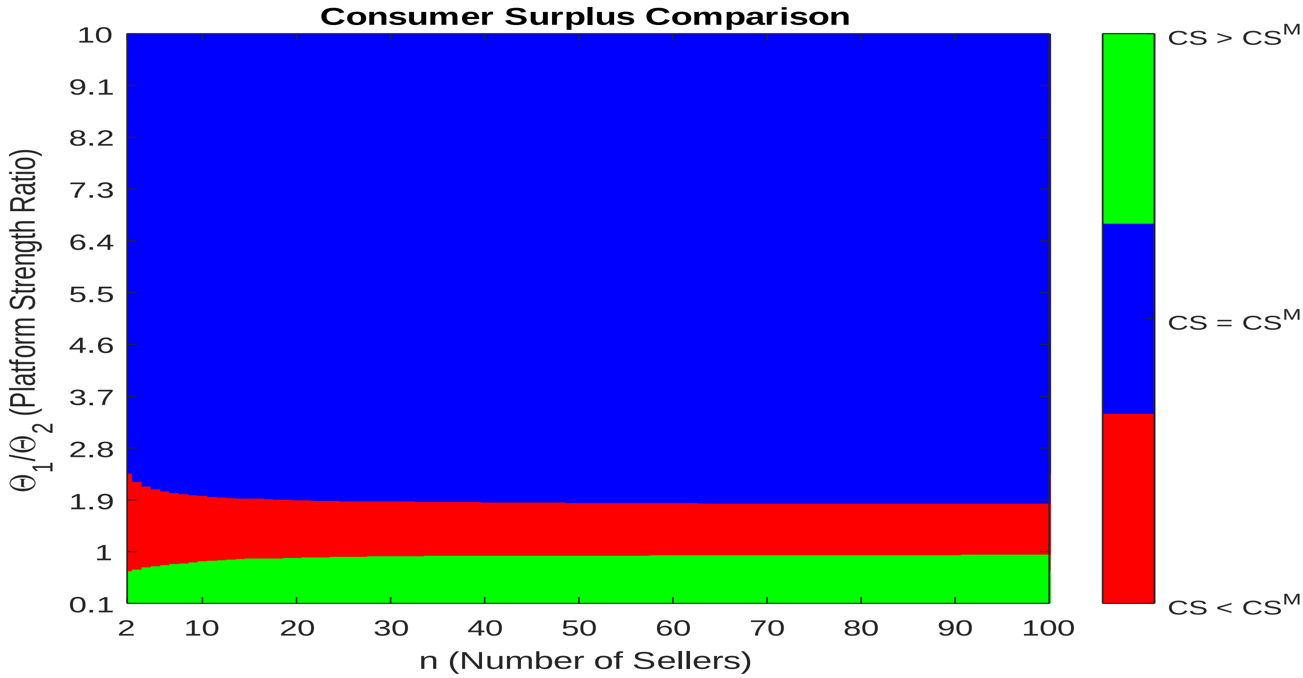

Figure 3: Comparison between CS and CS^M when $\gamma = 0.3$

Proof. Suppose Condition 5 holds.

Case I. $\frac{\Theta_H}{\Theta_L} > \phi_2$ or $\frac{\Theta_H}{\Theta_L} \in (1, \phi_1)$.

If Pi invests K to increase its platform strength from Θ_L to Θ_H, Proposition 2 implies that both S1 and S2 sell on Pi if Pk $(i, k \in \{1, 2\},\ i \neq k)$ does not invest, and each seller is indifferent between selling on Pi and selling on Pk if Pk invests K to increase its platform strength from Θ_L to Θ_H. Therefore, If Pi invests, then Pk's best response is to invest.

If Pi does not invest K to increase its platform strength from Θ_L to Θ_H, Proposition 2 implies that both S1 and S2 sell on Pk if Pk $(i, k \in \{1, 2\},\ i \neq k)$ invests K to increase its platform strength from Θ_L to Θ_H, and each seller is indifferent between selling on Pi and selling on Pk if Pk does not invest. Therefore, if Pi does not invest, Lemmas 3 and 6, and Proposition 2 imply that Pk's profit is: (i) $\Theta_H \left[M_{k1} + M_{k2} \right] - 2F - K$ if Pk invests; and (ii) $\frac{\Theta_L \left[\widetilde{\Delta}_{kj} \right]^2}{8\, b_j^S}$ if Pk does not invest. Condition 5 implies that if Pi does not invest, then Pk's best response is to invest.

Therefore, in equilibrium, both P1 and P2 invest to increase its platform strength from Θ_L to Θ_H, both platforms commit not to enter, and each seller is indifferent between selling on P1 and selling on P2.

Case II. $\frac{\Theta_H}{\Theta_L} \in (\phi_1, \phi_2)$.

If P2 invests to increase its platform strength from Θ_L to Θ_H, Proposition 2 implies that S1 sells on P2 whereas S2 sells on P1 if P1 does not invest, and each seller is indifferent between selling on P1 and selling on P2 if P1 invests. Proposition 2 also implies that P1 commits not to enter, regardless of its investment decision. Therefore, if P2 invests, Lemma 3 and Proposition 2 imply that P1's profit is: (i) $\frac{\Theta_L \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}$ if P1 does not invest; and (ii)

$\frac{\Theta_H [\widetilde{\Delta}_{12}]^2}{8\, b_2^S} - K$ if P1 invests.[1] Condition 5 ensures that $\frac{\Theta_H [\widetilde{\Delta}_{12}]^2}{8\, b_2^S} - K > \frac{\Theta_L [\widetilde{\Delta}_{12}]^2}{8\, b_2^S}$. Therefore, if P2 invests, then P1 invests.

If P1 invests to increase its platform strength from Θ_L to Θ_H, Proposition 2 implies that S1 sells on P1 whereas S2 sells on P2 if P2 does not invest, and each seller is indifferent between selling on P1 and selling on P2 if P2 invests. Proposition 2 also implies that P2 commits not to enter, regardless of its investment decision. Therefore, if P1 invests, Lemma 3 and Proposition 2 imply that P2's profit is: (i) $\frac{\Theta_L [\widetilde{\Delta}_{22}]^2}{8\, b_2^S}$ if P2 does not invest; and (ii) $\frac{\Theta_H [\widetilde{\Delta}_{22}]^2}{8\, b_2^S} - K$ if P2 invests. (13) and Condition 5 ensures that $\frac{\Theta_H [\widetilde{\Delta}_{22}]^2}{8\, b_2^S} - K > \frac{\Theta_L [\widetilde{\Delta}_{22}]^2}{8\, b_2^S}$. Therefore, if P1 invests, then P2 invests.

If P2 does not invest to increase its platform strength from Θ_L to Θ_H, Proposition 2 implies that S1 sells on P1 whereas S2 sells on P2 if P1 invests, and each seller is indifferent between selling on P1 and selling on P2 if P1 does not invest. Proposition 2 also implies that P1 makes no commitment if P1 invests whereas P1 commits not to enter if P1 does not invest. Therefore, if P2 does not invest, Lemmas 3 and 6, and Proposition 2 imply that P1's profit is: (i) $\Theta_H\, M_{11} - F - K$ if P1 invests; and (ii) $\frac{\Theta_L [\widetilde{\Delta}_{11}]^2}{8\, b_1^S}$ if P1 does not invest. Condition 5 ensures that $\Theta_H\, M_{11} - F - K > \frac{\Theta_L [\widetilde{\Delta}_{11}]^2}{8\, b_1^S}$. Therefore, if P2 does not invest, then P1 invests.

If P1 does not invest to increase its platform strength from Θ_L to Θ_H, Proposition 2 implies that S1 sells on P2 whereas S2 sells on P1 if P2 invests, and each seller is indifferent between selling on P1 and selling on P2 if P2 does not invest. Proposition 2 also implies that P2 makes no commitment if P2 invests whereas P2 commits not to enter if P2 does not invest. Therefore, if P1 does not invest, Lemmas 3 and 6, and Proposition 2 imply that P2's profit is: (i) $\Theta_H\, M_{21} - F - K$ if P2 invests; and (ii) $\frac{\Theta_L [\widetilde{\Delta}_{21}]^2}{8\, b_1^S}$ if P2 does not invest. Condition 5 ensures that $\Theta_H\, M_{21} - F - K > \frac{\Theta_L [\widetilde{\Delta}_{21}]^2}{8\, b_1^S}$. Therefore, if P1 does not invest, then P2 invests.

Consequently, in equilibrium, both P1 and P2 invest to increase its platform strength from Θ_L to Θ_H, both platforms commit not to enter, and each seller is indifferent between selling on P1 and selling on P2. ■

In the monopolistic platform setting, the platform (e.g., P1) does not invest to increase its strength in equilibrium because both sellers will sell on the platform regardless. Therefore, Proposition 1 shows that the consumer surplus in the monopolistic platform setting is:

$$CS^M \;=\; \Theta_L\, [\varsigma_{11} + \varsigma_{12}]\,, \tag{223}$$

where ς_{kj} is given by (8).

(13) and Proposition 8 imply that the consumer surplus in the platform competition setting is:

$$CS = \frac{\Theta_H}{2\, b_1^S} \left[\frac{\widetilde{\Delta}_{11}}{4} \right]^2 + \frac{\Theta_H}{2\, b_2^S} \left[\frac{\widetilde{\Delta}_{22}}{4} \right]^2 .$$

[1] This is the case because $\widetilde{\Delta}_{11} = \widetilde{\Delta}_{12}$ and $b_1^S = b_2^S$.

Figure 4 illustrates numerical solutions for settings where for $j \in \{1,2\}$, $\eta_j = 0.4$, $\beta_j^S = \beta_j^P = 0.5$ (so that $\Omega_j \in (0,1)$), $\alpha_j = 10$, $c_1^P = 1$, $c_2^P = 2$, $c_1^S = 3$, $c_2^S = 4$, $\frac{\Theta_H}{\Theta_L} \in (1, 10]$, and $\theta \in [0.1, 10]$.[2] The X-axis represents relative popularity: $\theta < 1$ indicates that the third-party seller's product is more popular, $\theta > 1$ indicates that the platform's product is more popular, and $\theta = 1$ indicates equal popularity. The Y-axis represents the potential increase in strength a platform can achieve through investment. In this setting, platforms can replicate sellers' products at a cost lower than the sellers' production costs (i.e., $c_k^P < c_j^S$). Additionally, the incumbent platform can imitate the seller's product at a lower cost than the entrant platform (i.e., $c_1^P < c_2^P$).

Figure 4 demonstrates that when endogenous platform strength is taken into account, increased platform competition can reduce consumer surplus unless platforms can increase their platform strengths significantly (i.e,. $\frac{\Theta_H}{\Theta_L}$ is sufficiently pronouned.)

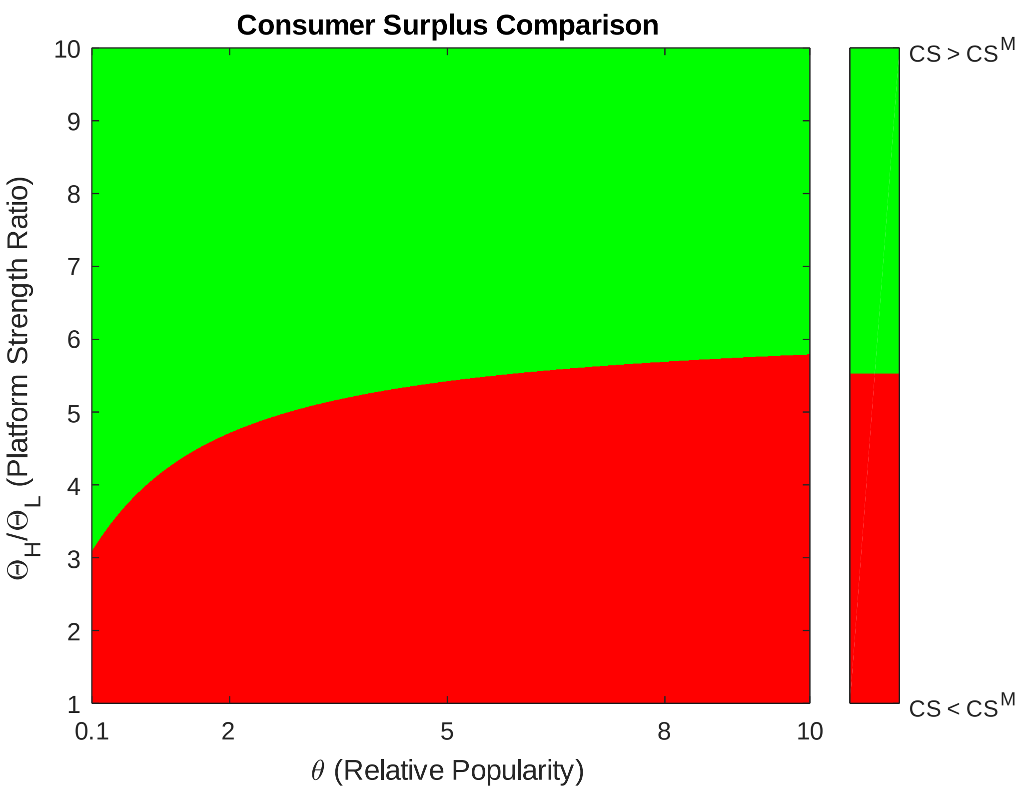

Figure 4: Comparison between CS and CS^M in the setting of endogenous platform strength

Figure 5 illustrates numerical solutions for settings where for $j \in \{1,2\}$, $\eta_j = 0.4$, $\beta_j^S = \beta_j^P = 0.5$ (so that $\Omega_j \in (0,1)$), $\alpha_j = 10$, $c_1^P = 5$, $c_2^P = 6$, $c_1^S = 3$, $c_2^S = 4$, $\frac{\Theta_H}{\Theta_L} \in (1, 10]$, and $\theta \in [0.1, 10]$. In this setting, platforms can replicate sellers' products at a cost higher than the sellers' production costs (i.e., $c_k^P > c_j^S$). Additionally, the incumbent platform can

[2] To align with the main analysis, the parameters η_j, β_j^S, β_j^P, α_j, c_j^P, c_j^S are chosen such that $\Omega_j \in (0,1)$, $c_2^S > c_1^S$, $\Delta_{kj} > 0$, $\overline{\Delta}_{kj} > 0$, and $\widetilde{\Delta}_{kj} > 0$ for all values of $\theta \in [0.1, 10]$.

imitate the seller's product at a lower cost than the entrant platform (i.e., $c_1^P < c_2^P$).

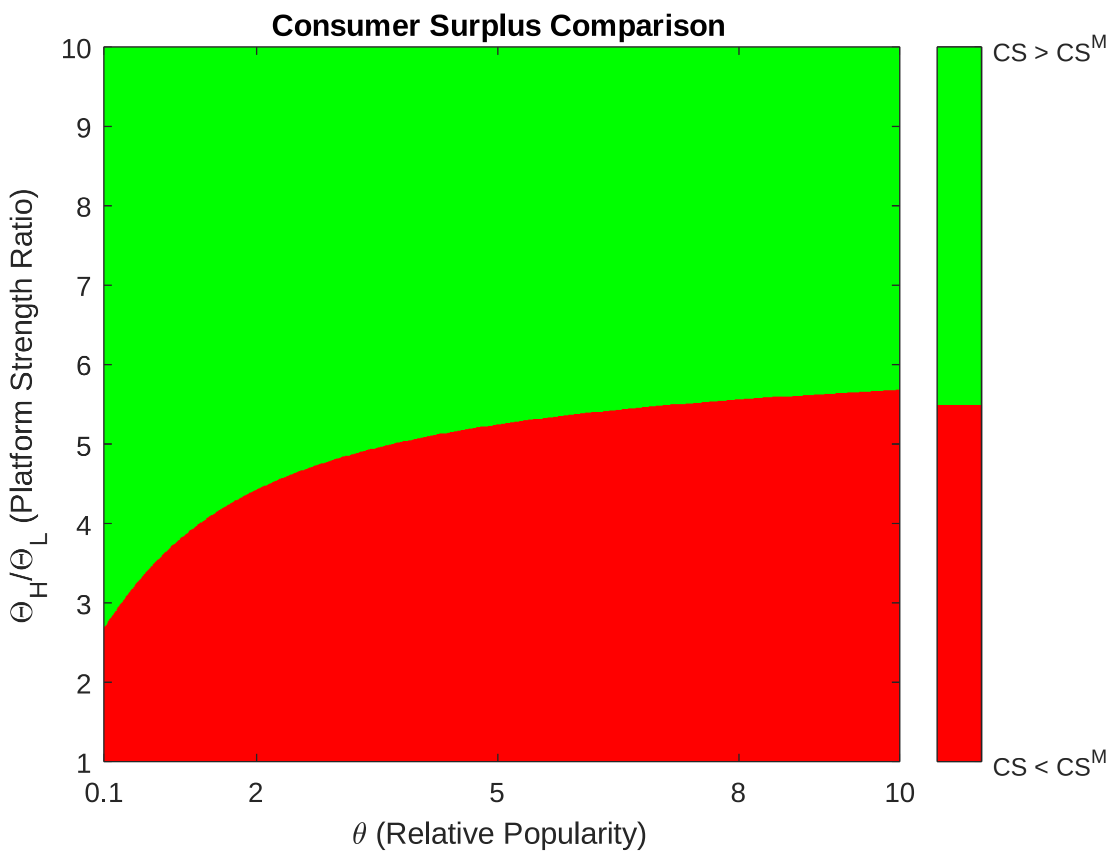

Figure 5: Comparison between CS and CS^M where $c_k^P > c_j^S$

Figure 6 illustrates numerical solutions for settings where for $j \in \{1, 2\}$, $\eta_j = 0.4$, $\beta_j^S = \beta_j^P = 0.5$ (so that $\Omega_j \in (0, 1)$), $\alpha_j = 10$, $c_1^P = 2$, $c_2^P = 1$, $c_1^S = 3$, $c_2^S = 4$, $\frac{\Theta_H}{\Theta_L} \in (1, 10]$, and $\theta \in [0.1, 10]$. In this setting, platforms can replicate sellers' products at a cost lower than the sellers' production costs (i.e., $c_k^P < c_j^S$). Additionally, the entrant platform can imitate the seller's product at a lower cost than the incumbent platform (i.e., $c_2^P < c_1^P$).

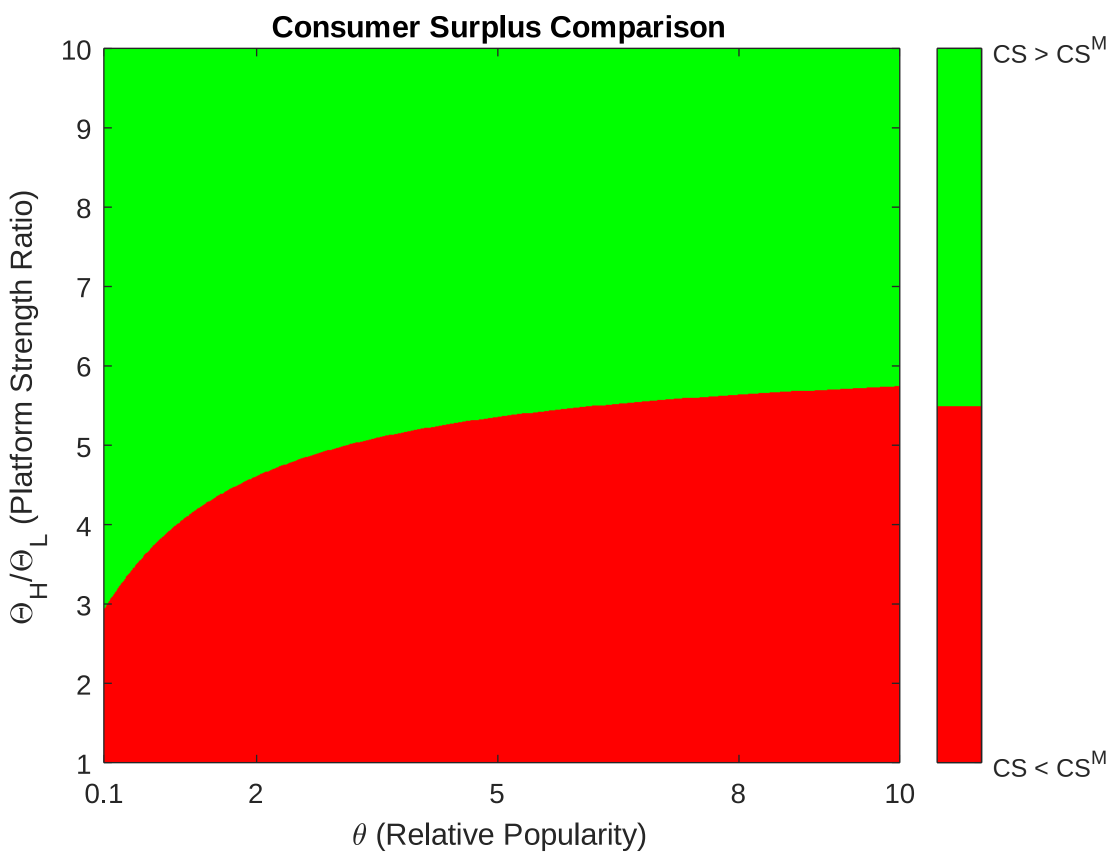

Figure 6: Comparison between CS and CS^M where $c_2^P < c_1^P$

I further explore the scenario where platforms compete on platform strength in the initial state, i.e., each platform endogenously decides its platform strength before committing to their selling capabilities. Specifically, I consider a game in which platforms simultaneously choose their platform strengths. Then platforms simultaneously choose to either commit not to act as sellers or make no such commitment. After the platform strengths and commitments are specified, S1 and S2 choose the platform on which they will sell (simultaneously and independently). Next, a platform that made no commitment will make its entry decision (i.e., whether to enter and which market to enter). Then P1 and P2 simultaneously set their per-unit commissions. Finally, each active seller sets its profit-maximizing price for its product.

Proposition 2 suggests that in equilibrium, both platforms opt for a higher level of platform strength compared to what a monopolistic platform would choose. This occurs because increased platform competition compels each platform to increase its strength to attract sellers. If platforms were to choose differing strengths, the one with the lower strength would be motivated to enhance it to attract more sellers. Therefore, each platform matches the other's strength in equilibrium. Furthermore, Proposition 2 indicates that platforms commit not to enter the seller market when they have equivalent platform strengths. Proposition 2, (233), and (223) imply that consumer surplus under Monopoly Platform (MP) is $CS^M = \Theta\,[\,\varsigma_{P1} + \varsigma_{P2}\,]$, while consumer surplus under Platform Competition (PC) is $CS = \widehat{\Theta}\left[\frac{\left(\frac{\tilde{\Delta}_{k1}}{4}\right)^2}{2\,\beta_1^S[1-\Omega_1]} + \frac{\left(\frac{\tilde{\Delta}_{i2}}{4}\right)^2}{2\,\beta_2^S[1-\Omega_2]}\right]$, where ς_{kj} is defined in (8), Θ represents the strength level chosen by the monopoly platform under MP, $\widehat{\Theta}$ represents the strength level chosen by each platforms under PC, and $\widehat{\Theta} > \Theta$. Therefore, $CS \lesseqgtr CS^M$ if $\frac{\widehat{\Theta}}{\Theta} \lesseqgtr \frac{\varsigma_{P1}+\varsigma_{P2}}{\frac{\left[\frac{\tilde{\Delta}_{k1}}{4}\right]^2}{2\,\beta_1^S[1-\Omega_1]} + \frac{\left[\frac{\tilde{\Delta}_{i2}}{4}\right]^2}{2\,\beta_2^S[1-\Omega_2]}}$, where $\frac{\widehat{\Theta}}{\Theta}$ represents the consumers' benefits from increased platform strength, and $\frac{\varsigma_{P1}+\varsigma_{P2}}{\frac{\left[\frac{\tilde{\Delta}_{k1}}{4}\right]^2}{2\,\beta_1^S[1-\Omega_1]} + \frac{\left[\frac{\tilde{\Delta}_{i2}}{4}\right]^2}{2\,\beta_2^S[1-\Omega_2]}}$ indicates the consumers' loss from higher prices due to reduced downstream competition.

4 Discriminating commitments

In this section, I consider the case where platforms can make discriminatory commitments, i.e., a platform might commit to not entering one seller's market while making no such promise to another seller.

Proposition 9. *Suppose platform entry is feasible (Condition FS holds), a third-party seller benefits more from no competition than from reduced competition (Assumption BC holds), and platforms can make discriminating commitments. Further suppose* $\frac{\Theta_1}{\Theta_2} \geq 1$ *and* $c^P_{1j} < c^P_{2j}$*. Then in equilibrium: (i) if* $\frac{\Theta_1}{\Theta_2} > \phi_2$*, P1 makes no commitment to both sellers, and both S1 and S2 sell on P1; (ii) if* $\frac{\Theta_1}{\Theta_2} \in (\phi_1, \phi_2)$*, P1 commits not to enter S2's market but makes no commitment for S1, and both sellers sell on P1; (iii) if* $\frac{\Theta_1}{\Theta_2} \in (1, \phi_1)$*, P1 commits not to enter both sellers' markets, and both S1 and S2 sell on P1; and (iv) if* $\frac{\Theta_1}{\Theta_2} = 1$*, both platforms commit not to enter either seller's market, and each seller is indifferent between selling on P1 and selling on P2.*

Proof. Condition FS ensures that each platform enters each seller's market if the platform makes no commitment. Since S1 and S2 sell independent products, S1's choice of platform is independent of S2's choice of platform.

Case I. $c^P_{1j} < c^P_{2j}$.

It can be shown that:

$$\phi_2 \; > \; \phi_1 \; > \; \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 \; > \; \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2 \; > \; 1 \;\text{ if }\; c^P_{1j} \; < \; c^P_{2j}. \tag{224}$$

First suppose $\frac{\Theta_1}{\Theta_2} > \phi_2$. Lemmas 7 - 10 and (224) imply that both S1 and S2 sells on P1, regardless of the platforms' commitments. Condition FS ensures P1 enters each seller's market. Therefore, Lemmas 3 and 6 imply that: (i) P1's profit is $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\Theta_1[\widetilde{\Delta}_{11}]^2}{8 b^S_1} + \frac{\Theta_1[\widetilde{\Delta}_{12}]^2}{8 b^S_2}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1[\widetilde{\Delta}_{1j}]^2}{8 b^S_j}$, i.e., P1 secures more profit by making no commitment than by committing not to enter. Therefore, in equilibrium, P1 makes no commitment to both sellers, and both S1 and S2 sell on P1 if $\frac{\Theta_1}{\Theta_2} > \phi_2$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in (\phi_1, \phi_2)$. Lemmas 7 - 10 and (224) imply that S1 sells on P1, regardless of the platforms' commitments. Condition FS and Lemmas 3 and 6 imply that P1 secures more profit by making no commitment to S1 than by committing not to enter S1's market. If P2 makes no commitment, Lemmas 8, 9 and (224) imply that S2 sells on P1. If P2 commits not to enter, Lemmas 7 and 10 imply that S2: (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter S2's market than by making no commitment to S2. Consequently, in equilibrium, P1 makes no commitment to S1 but commits to not entering S2's market, and both sellers sell on P1 if $\frac{\Theta_1}{\Theta_2} \in (\phi_1, \phi_2)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2, \phi_1 \right)$. If P2 makes no commitment, Lemmas 8, 9 and (224) imply that S1 and S2 both sell on P1, regardless of P1's commitment. Condition FS and Lemmas 3 and 6 imply that P1 secures more profit by making no commitment to both sellers than by committing not to enter each seller's market in this case. If P2 commits not to enter, Lemmas 7, 10 and (224) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter Sj's market; and (ii) sells on P2 if P1 makes no commitment to Sj. Therefore, P1 secures more profit by committing not to enter Sj's market than by making no commitment to Sj in this case. If P1 commits not to enter, Lemmas 7, 9 and (224) imply that both S1 and S2 sell on P1, regardless of P2's commitment. If P1 makes no commitment, Lemmas 8, 10, and (224) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P2 makes no commitment to Sj; and (ii) sells on P2 if P2 commits not to enter Sj's market. Therefore, P2 secures more profit by committing not to enter Sj's market than by making no commitment to Sj in this case. Consequently, in equilibrium, both P1 and P2 commit not to enter each seller's market, and both S1 and S2 sell on P1, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2, \phi_1 \right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2 \right)$. If P2 makes no commitment, Lemmas 8, 9, and (224) imply that S1 sells on P1 and S2: (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Condition FS and Lemmas 3 and 6 imply that P1 makes no commitment to S1 but commits not to enter S2's market in this case. If P2 commits not to enter, Lemmas 7, 10, and (224) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 commits not to enter each seller's market in this case. If P1 commits not to enter, Lemmas 7, 9, and (224) imply that both S1 and S2 sell on P1, regardless of P2's commitment. If P1 makes no commitment, Lemmas 8, 10, and (224) imply that S2 sells on P2 and S1: (i) sells on P1 if P2 makes no commitment; and (ii) sells on P2 if P2 commits not to enter. Therefore, P2 makes no commitment to S2 but commits not to enter S1's market in this case. Consequently, in equilibrium, both P1 and P2 commit not to enter each seller's market, and both S1 and S2 sell on P1, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2 \right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2$. If P2 makes no commitment, Lemmas 8, 9, and (224) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 commits not to enter each seller's market in this case. If P2 commits not to enter, Lemmas 7, 10, and (224) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 commits not to enter each seller's market in this case. If P1 commits not to enter, Lemmas 7, 9, and (224) imply that both S1 and S2 sell on P1, regardless of P2's commitment. Consequently, in equilibrium, P1 commits not to enter each seller's market, and both S1 and S2 sell on P1, if $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2$.

Finally, suppose $\frac{\Theta_1}{\Theta_2} = 1$. Lemma 7 implies that if both platforms commit not to enter Sj's market, then Sj ($j \in \{1, 2\}$) is indifferent between selling on P1 and selling on P2.

Lemma 8 implies that if platforms both make no commitment to Sj, then Sj sells on P2. Lemma 9 implies that if P1 commits not to enter Sj's market and P2 makes no commitment to Sj, then Sj sells on P1. Lemma 10 implies that if P1 makes no commitment to Sj and P2 commits to no entry Sj's market, then Sj sells on P2. If P2 makes no commitment, Sj: (i) sells on P1 if P1 commits not to enter Sj's market; and (ii) sells on P2 if P1 makes no commitment to Sj. If P2 commits not to enter Sj's market, Sj: (i) is indifferent between selling on P1 and selling on P2 if P1 commits not to enter Sj's market; and (ii) sells on P2 if P1 makes no commitment to Sj. If P1 commits not to enter Sj's market, Sj: (i) is indifferent between selling on P1 and selling on P2 if P2 commits not to enter Sj's market; and (ii) sells on P1 if P2 makes no commitment to Sj. Consequently, in equilibrium, both platforms commit not to enter each seller's market, and each seller is indifferent between selling on P1 and selling on P2, if $\frac{\Theta_1}{\Theta_2} = 1$. ■

Proposition 10. *Suppose platform entry is feasible (Condition FS holds), a third-party seller benefits more from no competition than from reduced competition (Assumption BC holds), and platforms can make discriminating commitments. Further suppose P faces a competing platform $\widetilde{P}$ that is a weaker seller than P (i.e., $\frac{\tilde{c}_j^P}{c_j^P} > 1$). Then $CS < CS^M$ unless $\widetilde{P}$'s relative platform strength is sufficiently pronounced.*

Proof. (1) implies that if Sj sells on Pk and competes against Pk ($j, k \in \{1, 2\}$), then:

$$p_{kj}^S = \frac{q_{kj}^P}{\eta_j} - \frac{\theta_j \, \alpha_j}{\eta_j} + \frac{\beta_j^P \, p_{kj}^P}{\eta_j}. \tag{225}$$

(2) and (225) imply that:

$$p_{kj}^P = \frac{\eta_j \, q_{kj}^S + \beta_j^S \, q_{kj}^P - \alpha_j \left[\eta_j + \beta_j^S \, \theta_j \right]}{\left[\eta_j \right]^2 - \beta_j^S \, \beta_j^P}. \tag{226}$$

(225) and (226) imply that:

$$p_{kj}^S = \frac{\eta_j \, q_{kj}^P + \beta_j^P \, q_{kj}^S - \alpha_j \left[\theta_j \, \eta_j + \beta_j^P \right]}{\left[\eta_j \right]^2 - \beta_j^S \, \beta_j^P}. \tag{227}$$

Lemma 6 implies that if Sj competes against Pk, then:

$$q_{kj}^{*S} = \frac{Q_{kj}^{*S}}{\Theta_k} = \frac{\left[2 + \Omega_j \right] \Delta_{kj}}{8 + \Omega_j} \text{ and } q_{kj}^{*P} = \frac{Q_{kj}^{*P}}{\Theta_k} = \frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \left[2 + \Omega_j \right] \Delta_{kj}}{2 \beta_j^S \left[8 + \Omega_j \right]}. \tag{228}$$

(226), (227), and (228) imply that if S1 competes against Pk and S2 competes against Pi ($k, i \in \{1, 2\}$), then consumer surplus is:

$$CS = \Theta_k \, \varsigma_{k1} + \Theta_i \, \varsigma_{i2}, \tag{229}$$

where for $j \in \{1, 2\}$, ς_{kj} is given by (8).

It can be shown that:

$$\frac{\partial \varsigma_{kj}}{\partial c^P_{kj}} \;<\; 0. \tag{230}$$

(3) implies that if Sj sells on Pk ($j, k \in \{1, 2\}$) and faces no competition, then:

$$q^S_{kj} \;=\; A_j - b^S_j\, p^S_{kj} \;\;\Leftrightarrow\;\; p^S_{kj} = \frac{A_j}{b^S_j} - \frac{q^S_{kj}}{b^S_j}. \tag{231}$$

Lemma 3 implies that if Sj sells on Pk ($k, i \in \{1, 2\}$) and faces no competition, then:

$$q^{*S}_{kj} \;=\; \frac{Q^{*S}_j}{\Theta_k} \;=\; \frac{\widetilde{\Delta}_{kj}}{4}. \tag{232}$$

(231) and (232) imply that if S1 sells on Pk, S2 sells on Pi ($k, i \in \{1, 2\}$), and each seller faces no competition, then consumer surplus is:

$$CS \;=\; \frac{\Theta_k}{2\,\beta^S_1\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{k1}}{4}\right]^2 + \frac{\Theta_i}{2\,\beta^S_2\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{i2}}{4}\right]^2. \tag{233}$$

(226), (227), (228), (8), (231), and (232) imply that if S1 competes against Pk and S2 sells on Pi and faces no competition ($k, i \in \{1, 2\}$), then consumer surplus is:

$$CS \;=\; \Theta_k\,\varsigma_{k1} + \frac{\Theta_i}{2\,\beta^S_2\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{i2}}{4}\right]^2. \tag{234}$$

(12) and (8) imply that:

$$\varsigma_{kj} \;>\; \frac{1}{2\,\beta^S_j\,[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{kj}}{4}\right]^2. \tag{235}$$

Proposition 1 implies that in the benchmark setting (MP), the monopoly platform makes no commitment to either sellers in equilibrium. As a result, each seller competes against P under MP. Therefore, (229) implies that:

$$CS^M \;=\; \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2}. \tag{236}$$

First suppose P faces a competing platform $\widetilde{\text{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$) under PC, where $\widetilde{\Theta}$ denotes $\widetilde{\text{P}}$'s platform strength. (230) and $\frac{\widetilde{c}^P_j}{c^P_j} > 1$ imply that:

$$\frac{\varsigma_{Pj}}{\varsigma_{\widetilde{P}j}} \;>\; 1. \tag{237}$$

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} > \max\left\{\frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\}$.

Because $\frac{\widetilde{\Theta}}{\Theta} > \phi_{\widetilde{P}2}$, Proposition 9 implies that each seller competes against $\widetilde{\mathrm{P}}$ under PC. Therefore, (229) implies that:

$$CS \;=\; \widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \widetilde{\Theta}\,\varsigma_{\widetilde{P}2}. \tag{238}$$

Because $\frac{\widetilde{\Theta}}{\Theta} > \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}$, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \widetilde{\Theta}\,\varsigma_{\widetilde{P}2} > \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2}$. Therefore, (236) and (238) imply that $CS > CS^M$ in this case.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \min\left\{\frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\}\right)$.

First suppose $\phi_{\widetilde{P}1} < \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}$. If $\frac{\widetilde{\Theta}}{\Theta} \in \left(\phi_{\widetilde{P}1}, \min\left\{\frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\}\right)$, then $\frac{\widetilde{\Theta}}{\Theta} \in \left(\phi_{\widetilde{P}1}, \phi_{\widetilde{P}2}\right)$. Proposition 9 implies that S1 competes against $\widetilde{\mathrm{P}}$ whereas S2 sells on $\widetilde{\mathrm{P}}$ and faces no competition under PC. Therefore, (234) implies that:

$$CS \;=\; \widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \frac{\widetilde{\Theta}}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2. \tag{239}$$

Because $\frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}$, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \widetilde{\Theta}\,\varsigma_{\widetilde{P}2} < \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2}$. (235) implies that $\frac{1}{2\,\beta_2^S[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2 < \varsigma_{\widetilde{P}2}$. Therefore, (236) and (239) imply that $CS < CS^M$ in this case. If $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \phi_{\widetilde{P}1}\right)$, Proposition 9 implies that each seller sells on $\widetilde{\mathrm{P}}$ and faces no competition under PC. Therefore, (233) implies that:

$$CS \;=\; \frac{\widetilde{\Theta}}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}1}}{4}\right]^2 + \frac{\widetilde{\Theta}}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2. \tag{240}$$

(235) implies that $\frac{\widetilde{\Theta}}{2\,\beta_j^S[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}j}}{4}\right]^2 < \widetilde{\Theta}\,\varsigma_{\widetilde{P}j}$. Because $\frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}$, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \widetilde{\Theta}\,\varsigma_{\widetilde{P}2} < \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2}$. Therefore, (236) and (240) imply that $CS < CS^M$ in this case.

Next suppose $\phi_{\widetilde{P}1} \geq \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}$. It can be shown that $\phi_{\widetilde{P}2} > \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}$. Therefore, $\min\left\{\frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\} = \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}$. Because $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \frac{\varsigma_{P1}+\varsigma_{P2}}{\varsigma_{\widetilde{P}1}+\varsigma_{\widetilde{P}2}}\right)$, $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \phi_{\widetilde{P}1}\right)$. Proposition 9 implies that each seller sells on $\widetilde{\mathrm{P}}$ and faces no competition under PC. Therefore, consumer surplus is given by (240). Therefore, (236) and (240) imply that $CS < CS^M$ in this case. Next suppose P faces a symmetric competing platform $\widetilde{\mathrm{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta} = 1$) under PC. Proposition 9 implies that each seller is indifferent between selling on P and selling on $\widetilde{\mathrm{P}}$ and each seller faces no competition under PC. Therefore, (233) implies that:

$$\begin{aligned} CS \;=\; & \frac{1}{2}\,\frac{\widetilde{\Theta}}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}1}}{4}\right]^2 + \frac{1}{2}\,\frac{\Theta}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{P1}}{4}\right]^2 \\ & + \frac{1}{2}\,\frac{\widetilde{\Theta}}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2 + \frac{1}{2}\,\frac{\Theta}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{P2}}{4}\right]^2. \end{aligned} \tag{241}$$

(235) implies that $\frac{\Theta}{2\,\beta_j^S[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{Pj}}{4}\right]^2 < \Theta\,\varsigma_{Pj}$ and $\frac{\widetilde{\Theta}}{2\,\beta_j^S[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}j}}{4}\right]^2 < \widetilde{\Theta}\,\varsigma_{\widetilde{P}j}$. Because $\frac{\widetilde{\Theta}}{\Theta} = 1$, (241) implies that

$$CS < \frac{1}{2}\,\widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \frac{1}{2}\,\Theta\,\varsigma_{P1} + \frac{1}{2}\,\widetilde{\Theta}\,\varsigma_{\widetilde{P}2} + \frac{1}{2}\,\Theta\,\varsigma_{P2} = \frac{1}{2}\,\Theta\left[\,\varsigma_{\widetilde{P}1} + \varsigma_{P1} + \varsigma_{\widetilde{P}2} + \varsigma_{P2}\right]$$

$$< \frac{1}{2}\,\Theta\left[\varsigma_{P1} + \varsigma_{P1} + \varsigma_{P2} + \varsigma_{P2}\right] = \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2} = CS^M. \tag{242}$$

The first inequality in (242) reflects (237) and the last inequality in (242) reflects (236).

Finally, suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_{P1}}, 1\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,1, \phi_1\,)$. Proposition 9 implies that each seller sells on P and faces no competition under PC. Therefore, (233) implies that:

$$CS \;=\; \frac{\Theta}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{P1}}{4}\right]^2 + \frac{\Theta}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{P2}}{4}\right]^2. \tag{243}$$

(235) implies that $\frac{1}{2\,\beta_j^S[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{Pj}}{4}\right]^2 < \varsigma_{Pj}$ for $j \in \{1,2\}$. Therefore, (236) and (243) imply that $CS < CS^M$ in this case.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_{P2}}, \frac{1}{\phi_{P1}}\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,\phi_1, \phi_2\,)$. Proposition 9 implies that S1 competes against P whereas S2 sells on P and faces no competition under PC. Therefore, (234) implies that:

$$CS \;=\; \Theta\,\varsigma_{P1} + \frac{\Theta}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{P2}}{4}\right]^2. \tag{244}$$

Therefore, (235), (236), and (244) imply that $CS < CS^M$ in this case. ■